Comparison Methods for Testing Series Name Solutions

Determine if the given series converges or diverges. Indicate the method that you use and show the work that leads to your conclusion.

1) $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \frac{1}{e}\left(\frac{1}{e}\right)^{n-1}$

$\left|\frac{1}{e}\right| < 1$ so Geometric Series Converges

2) $\sum_{n=1}^{\infty} \frac{1}{e^n+1} \leq \sum \left(\frac{1}{e}\right)^n$

So by direct comparison to #1, the series converges

3) $\sum_{n=1}^{\infty} \frac{1}{e^n-3}$ Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{1}{e^n-3}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{e^n}{e^n-3} = 1$$

So series converges

4) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^e = \sum_{n=1}^{\infty} \frac{1}{n^e}$ p series with $p = e > 1$

So series converges

5) $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^e$ Limit Comparison Test Using $\left(\frac{1}{n}\right)^e$

$$\lim_{n\to\infty} \frac{\left(\frac{2}{n}\right)^e}{\left(\frac{1}{n}\right)^e} = 2$$

So series converges

6) $\sum_{n=1}^{\infty} \left(\frac{1}{n+2}\right)^e$ L.C. Test

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n+2}\right)^e}{\left(\frac{1}{n}\right)^e} = \left(\frac{n}{n+2}\right)^e = 1$$

Series Converges

7) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+5}}{\sqrt{n^5+7}}$ L.C. test with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{n^{3/2}}{n^{5/2}}$$

$$= \sum \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{\sqrt{n^3+5}}{\sqrt{n^5+7}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\sqrt{n^5+5n^2}}{\sqrt{n^5+7}} = 1$$

so series diverges by comparison with the Harmonic Series

8) $\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$ L.C. test with

$$\sum_{n=1}^{\infty} \frac{n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow \text{geometric series}$$

$r = \frac{1}{2}$

$$\lim_{n\to\infty} \frac{n+1}{n2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

∴ Series converges

9) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ geometric series

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \qquad r = \frac{2}{3} < 1$$

series converges

For exercises 10 and 11, prove that the series converges and find its sum.

10) $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$ geometric series with $|r|<1$

$$= \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^n + \frac{1}{3}\left(\frac{1}{3}\right)^n$$

$$\frac{\frac{2}{3}}{1-\frac{2}{3}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} = 2 + \frac{1}{2} = 2\frac{1}{2}$$

11) $\sum_{n=0}^{\infty} \frac{3}{n^2+4n+3} = \sum_{n=0}^{\infty} \frac{A}{n+3} + \frac{B}{n+1}$

$$An + A + Bn + 3B = 3$$

$$\left.\begin{array}{l} A+B=0 \\ A+3B=3 \end{array}\right\} \quad B = \frac{3}{2} \quad A = -\frac{3}{2}$$

$$\frac{3}{2}\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

$$\frac{3}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}\right.$$

$$\left. \ldots \cancel{\frac{1}{n}} - \frac{1}{n+2} + \cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right)$$

$$\frac{3}{2}\left(1 + \frac{1}{2} + \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{3}{n^2+4n+3} = \frac{9}{4}$$