Continuity and Differentiability (Solutions)

A function must be continuous in order to be differentiable. To prove continuity, prove that

$$\lim_{x \to a} f(x) = f(a)$$

To prove differentiability, prove that

$$\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$$

Remember that in the case of both limits, in order to exist the limit must be the same from the left side as from the right side.

1. $f(x) = \begin{cases} 3x^2 - 5, & x \le -1 \\ 2x^3, & x > -1 \end{cases}$ is continuous but not differentiable at $x = -1$

Because

$$\lim_{x \to -1^-} 3x^2 - 5 = -2$$

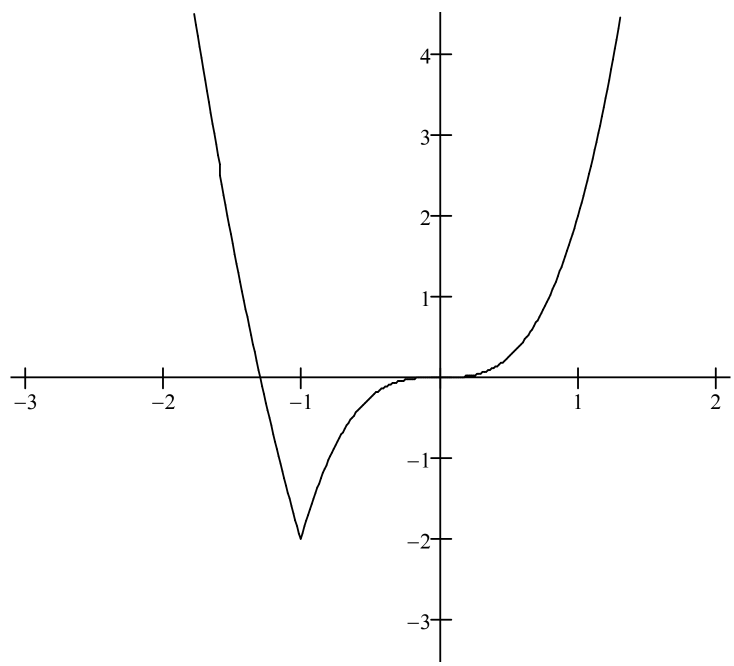
$$\lim_{x \to -1^+} 2x^3 = -2$$

$f(x)$ is continuous but since

$\lim_{x \to -1^-} 6x = -6$ and

$$\lim_{x \to -1^+} 6x^2 = 6$$

it is not differentiable at $x = -1$



2. $g(x) = \begin{cases} x^2 - 2x - 3, & x \le 2 \\ 2x - 7, & x > 2 \end{cases}$ is both continuous and differentiable at $x = 2$

Because

$\lim_{x \to 2^-} x^2 - 2x - 3 = -3$ and

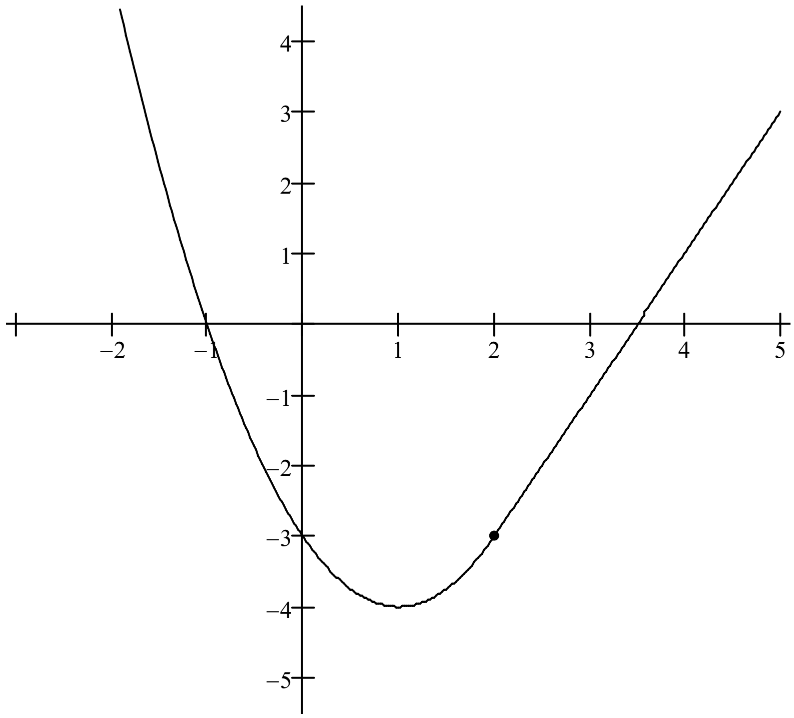
$\lim_{x \to 2^+} 2x - 7 = -3$

And since

$\lim_{x \to 2^-} 2x - 2 = 2$ and

$\lim_{x \to 2^+} 2 = 0$

$g(x)$ is differentiable at $x = 2$



3. $h(x) = \begin{cases} \cos(x), & x < 0 \\ x^2, & x \ge 0 \end{cases}$ is not continuous and therefore not differentiable at $x = 0$

Be careful with this one because while

$\lim_{x \to 0^-} -\sin(x) = 0$

and

$\lim_{x \to 0^+} 2x = 0$ indicate that $h(x)$ is differentiable,

remember that,

$\lim_{x \to 0^-} \cos(x) = 1$

And

$\lim_{x \to 0^+} x^2 = 0$

Since the function was never continuous, it can not be differentiable.