Integrating Factors Name______________________________

Show all your work

A first order linear differential equation can be written in the form
$\frac{dy}{dx} + P(x)y = Q(x)$ and solved by multiplying both sides by an integrating factor
$I(x) = e^{\int P(x)dx}$ transforming the left side of the equation into a product rule derivative of $I(x)y$.

Determine if the given differential equation is linear. If it is, solve for y using Integrating Factors when necessary.

1) $y' = x + 5y$

$y' - 5y = x$

$I(x) = e^{\int -5\,dx} = e^{-5x}$

$\underbrace{e^{-5x}y' - 5e^{-5x}y}_{(e^{-5x}y)'} = xe^{-5x}$

$e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$

u	dv
x	e^{-5x}
1	$-\frac{1}{5}e^{-5x}$
0	$\frac{1}{25}e^{-5x}$

$\boxed{y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}}$

2) $\frac{dy}{dx} + 2xy = 2x$

$I(x) = e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$

$(e^{x^2}y)' = 2xe^{x^2}$

$e^{x^2}y = \int 2xe^{x^2}\,dx$

$u = x^2$

$du = 2x\,dx$

$= \int e^u\,du$

$= e^u + C$

$e^{x^2}y = e^{x^2} + C$

$\boxed{y = 1 + Ce^{-x^2}}$

3) $xy' + y = \sqrt{x}$

$y' + \frac{1}{x}y = \frac{1}{\sqrt{x}}$

$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

back where we started

$xy' + y = \sqrt{x}$

$(xy)' = \sqrt{x}$

$xy = \frac{2}{3}x^{3/2} + C$

$y = \frac{2}{3}x^{1/2} + Cx^{-1}$

$\boxed{y = \frac{2}{3}\sqrt{x} + \frac{C}{x}}$

4) $1+e^x y = e^x y'$

$$y' - y = e^{-x}$$

$$I(x) = e^{-\int dx} = e^{-x}$$

$$e^{-x}y' - e^{-x}y = e^{-2x}$$

$$(e^{-x}y)' = e^{-2x}$$

$$e^{-x}y = \frac{-1}{2}e^{-2x} + C$$

$$\boxed{y = -\frac{1}{2}e^{-x} + Ce^{x}}$$

5) $\frac{y'}{x^2} - y = 1$

$$y' - x^2 y = x^2$$

$$I(x) = e^{-\int x^2 dx} = e^{-\frac{x^3}{3}}$$

$$e^{-\frac{x^3}{3}} - x^2 e^{-\frac{x^3}{3}} y = x^2 e^{-\frac{x^3}{3}}$$

$$\left(e^{-\frac{x^3}{3}} y\right) = \int x^2 e^{-\frac{x^3}{3}}\,dx$$

substitution

$$e^{-\frac{x^3}{3}} y = -e^{-x^3/3} + C$$

$$\boxed{y = Ce^{\frac{x^3}{3}} - 1}$$

6) $y' + 2y = 2e^x$

$$I(x) = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y' + 2e^{2x}y = 2e^{3x}$$

$$\left(e^{2x}y\right) = 2\int e^{3x}\,dx$$

$$e^{2x}y = \frac{2}{3}e^{3x} + C$$

$$\boxed{y = \frac{2}{3}e^{x} + Ce^{-2x}}$$

7) $(\sec x)y' + y = 1$

$$y' + (\cos x)y = \cos x$$

$$I(x) = e^{\int \cos x\,dx} = e^{\sin x}$$

$$e^{\sin x}y' + (\cos x)e^{\sin x}y = e^{\sin x}\cos x$$

$$\left(e^{\sin x}y\right) = \int e^{\sin x}\cos x\,dx$$

substitution

$$e^{\sin x}y = e^{\sin x} + C$$

$$\boxed{y = 1 + Ce^{-\sin x}}$$

Solve the initial value problem.

8) $t\dfrac{dy}{dt} + 2y = t^3, \quad y(1) = 0$

$\frac{dy}{dt} + \frac{2}{t} y' = t^2$

$I(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t}$

$= e^{\ln t^2} = t^2$

$t^2 \frac{dy}{dt} + 2t y' = t^4$

$(t^2 y)' = t^4$

$t^2 y = \frac{t^5}{5} + C$

$\boxed{y = \frac{t^3}{5} + \frac{C}{t^2}}$

$0 = \frac{1}{5} + \frac{C}{1}$

$C = -\frac{1}{5}$

$\boxed{y = \frac{t^3}{5} - \frac{1}{5t^2}}$

9) $\dfrac{dv}{dt} - 2tv = 3t^2 e^{t^2} \quad v(0) = 5$

$I(t) = e^{\int -2t\, dt} = e^{-t^2}$

$\frac{dv}{dt} e^{-t^2} - 2te^{-t^2} v = 3t^2$

$(e^{-t^2} v)' = 3t^2$

$e^{-t^2} v = t^3 + C$

$v = t^3 e^{t^2} + Ce^{t^2}$

$5 = 0e^0 + Ce^0$

$C = 5$

$\boxed{v(t) = t^3 e^{t^2} + 5e^{t^2}}$

10) For the given differential equation, find the three particular solutions using the three separate initial values given. Then sketch the three solution curves on the slope field below to show that each fit the slope field.

$y' = x + y, \quad y(0) = 0, \quad y(0) = 1, \quad y(-1) = 1$

$y' - y = x$

$I(x) = e^{-\int dx} = e^{-x}$

$e^{-x} - e^{-x}y = xe^{-x}$

$e^{-x}y = \int xe^{-x}\,dx = \underbrace{-xe^{-x} - e^{-x} + C}$

u	dv
x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

$e^{-x}y = -xe^{-x} - e^{-x} + C$

$\boxed{y = -x - 1 + Ce^{x}}$

For $y(0) = 0$ Blue

$\boxed{y = e^{x} - x - 1}$

For $y(0) = 1$ Green

$\boxed{y = 2e^{x} - x - 1}$

For $y(-1) = 1$ Red

$\boxed{y = e^{x+1} - x - 1}$