Logistic Growth Model

The exponential model for population growth($y = y_0e^{kt}$) assumes unlimited growth. This is realistic only for a short period of time when the initial population is small. A more realistic assumption is that the relative growth rate is positive but decreases as the population increases due to environmental or economic factors. In other words there is a maximum population M, the carrying capacity, that the environment is capable of sustaining in the long run. This growth rate can be given by the differential equation:

$$\frac{dP}{dt} = \frac{k}{M}P(M - P)$$ and can also be written as $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where P represents the function P(t), M is the carrying capacity, and *k* is a positive constant. The solution to this **logistic differential equation** is called the **logistic growth model**. Notice that the rate of growth is proportional to both P and (M – P). If P were to exceed M, the growth rate would be negative and the population would be decreasing.

State the solution to the differential equation above.

$P(t) =$

Example:

A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is $\frac{dP}{dt} = 0.0004P(250 - P)$, where time *t* is in years.

(a) State the formula for the gorilla population in terms of *t*

(b) about how long will it take for the gorilla population to reach the carrying capacity of the preserve?

1. A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$ where y is the proportion of the population($0 \leq y \leq 1$) that has heard the rumor at time *t*.
   (a) What proportion of the population has heard the rumor when it is spreading the fastest?
   (b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.
   (c) At what time t is the rumor spreading the fastest?

2. Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x, diffusion, *dx/dt*, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the differential equation $\frac{dx}{dt} = kx(N - x)$ where *N* is the size of the population. Suppose that *t* is time in days, $k = \frac{1}{250}$, and two people start a rumor at time $t = 0$ in a population of 1000 people.

   a) Find *x* as a function of *t*.

   b) When will half the population have heard the rumor?

   c) Show that the rumor is spreading the fastest at the time found in (b).

3. Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. the population $P(t)$ is increasing at a rate directly proportional to $800 - P$, where the constant of proportionality is k. (Note that this does not fit the general formula for logistic growth)

   a) If $P(0) = 500$, find $P(0)$ in terms of t and k.

   b) If $P(2) = 700$, find k.

   c) Find $\lim_{t \to \infty} P(t)$