Prove the following identities.

1. $\cos^2 x + \tan^2 x \cos^2 x = 1$

$\cos^2 x\ 1 + \tan x = 1$ Factor the $\cos^2 x$

$\cos^2 x(\sec^2 x) = 1$ Replace $1 + \tan^2 x$ with $\sec^2 x$

$\cos^2 x\left(\frac{1}{\cos^2 x}\right) = 1$ $\sec^2 x = \frac{1}{\cos^2 x}$

$1 = 1$

2. $2\cos\theta = \dfrac{\cos\theta \tan\theta + \sin\theta}{\tan\theta}$

$2\cos\theta = \dfrac{\cos\theta \frac{\sin\theta}{\cos\theta} + \sin\theta}{\frac{\sin\theta}{\cos\theta}}$ Write everything in terms of sine and cosine

$2\cos\theta = \dfrac{\cancel{\cos\theta} \frac{\sin\theta}{\cancel{\cos\theta}} + \sin\theta}{\frac{\sin\theta}{\cos\theta}}$ Cancel the cosines

$2\cos\theta = \dfrac{2\sin\theta}{\left(\frac{\sin\theta}{\cos\theta}\right)} \quad \frac{\cos\theta}{\sin\theta}$ Add the $\sin\theta$ terms and bring the $\frac{\sin\theta}{\cos\theta}$ out from the denominator and make it a multiplication by $\frac{\cos\theta}{\sin\theta}$

$2\cos\theta = 2\cancel{\sin\theta}\frac{\cos\theta}{\cancel{\sin\theta}}$ Cancel the sines

$2\cos\theta = 2\cos\theta$

4. $4+(\tan\sigma-\cot\sigma)^2=\sec^2\sigma+\csc^2\sigma$

$4+\tan^2\sigma-2\tan\sigma\cot\sigma+\cot^2\sigma=\sec^2\sigma+\csc^2\sigma$ Foil the parentheses

$4+\tan^2\sigma-2\cancel{\tan\sigma}\,\cancel{\cot\sigma}+\cot^2\sigma=\sec^2\sigma+\csc^2\sigma$ $\cot\sigma=\frac{1}{\tan\sigma}$ so

$\tan\sigma\cot\sigma=1$

$2+\tan^2\sigma+\cot^2\sigma=\sec^2\sigma+\csc^2\sigma$ Subtract the 2 from 4

$1+\tan^2\sigma+1+\cot^2\sigma=\sec^2\sigma+\csc^2\sigma$ Split the 2 into 2 1's to set up Pythagorean Identities

$\underbrace{(1+\tan^2\sigma)}_{\sec^2\sigma}+\underbrace{(1+\cot^2\sigma)}_{\csc^2\sigma}=\sec^2\sigma+\csc^2\sigma$ Pythagorean Identities

$\sec^2\sigma+\csc^2\sigma=\sec^2\sigma+\csc^2\sigma$