7-5: L'Hôpital's Rule

Guillaume De l'Hôpital
1661 - 1704

Johann Bernoulli
1667 - 1748

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L'Hôpital's Rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{0}{0}=\lim_{x\to 2}\frac{\frac{d}{dx}\left(x^2-4\right)}{\frac{d}{dx}\left(x-2\right)}=\lim_{x\to 2}\frac{2x}{1}=4$$

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \longleftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \longleftarrow \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \longleftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

$\rightarrow$

Example:

$$\lim_{x\to\infty}\frac{e^x-1}{2x} = \frac{\infty}{\infty}$$

Wait, what's this?

$\rightarrow$

L' Hôpital's rule can be used to evaluate other indeterminate forms besides $\frac{0}{0}$.

The following are also considered indeterminate:

$$\frac{\infty}{\infty} \qquad \infty \cdot 0 \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

The first one, $\frac{\infty}{\infty}$, can be evaluated just like $\frac{0}{0}$.

<u>The others must be changed to fractions first</u>.

$\rightarrow$

Example:

$$\lim_{x\to\infty}\frac{e^x-1}{2x} = \frac{\infty}{\infty}$$

First make sure it's indeterminate

Now we use L'Hopital's Rule

$$= \lim_{x\to\infty}\frac{e^x}{2} \qquad \frac{\infty}{2} = \infty$$

$\rightarrow$

Example:

$$\lim_{x\to\infty}\frac{2x}{e^x-1} = \frac{\infty}{\infty}$$

First make sure it's indeterminate

Now we use L'Hopital's Rule

$$=\lim_{x\to\infty}\frac{2}{e^x} \qquad \frac{2}{\infty}=0$$

$\rightarrow$

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$ ← This is indeterminate form $\infty-\infty$

If we find a common denominator and subtract, we get:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$ ← Now it is in the form $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$ ← L' Hôpital' s rule applied once.

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$ ← Fractions cleared. Still $\frac{0}{0}$

→

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

$$\lim_{x\to 1}\left(\frac{1}{1+1+\ln x}\right)$$ L' Hôpital again.

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

$$\frac{1}{2}$$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$

$\rightarrow$

End Behavior involves limits as x goes to infinity. Since we are dealing with indeterminate forms again, L'Hopital's Rule can help us.

Let's take a closer look at example 3 on page 403

EX 3 Find the end behavior of $y = xe^x$.

a) Left EB: $\lim_{x \to -\infty} xe^x = -\infty \bullet e^{-\infty} = \frac{-\infty}{e^{\infty}} = -\frac{\infty}{\infty}$

So let's rewrite this as a fraction:

$$\lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} = \frac{1}{-e^{-x}} = \frac{1}{-e^{-(-\infty)}} = \frac{1}{-e^{\infty}} = \frac{1}{-\infty} = 0$$

Apply L'Hopital's Rule

"Plug in" infinity

Simplify

Graph this on your calculator to see for yourself

b) Right EB: $\lim_{x \to +\infty} xe^x = \infty \bullet e^{\infty} = \infty$, so the right side goes up

So the end behavior is that the left side approaches $y = 0$ and the right side goes up.