# SI Geometry (Honors Edition)
## 4th Quarter

Rev. May, 2018

SAINT IGNATIUS COLLEGE PREPARATORY

SI

IHS

EST. 1855

# TABLE OF CONTENTS

# Unit A:

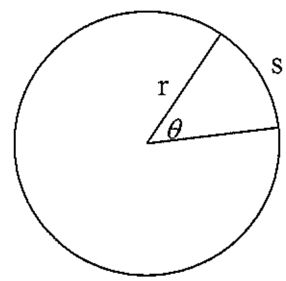
# Analytic Trigonometry

# Section A1—RADIANS

LEARNING OUTCOMES
- Express angles in radian measure
- Convert radian measure to degrees

Until now, degrees have been used to measure angles. In this section, another way to measure angles will be explored—***radians***.

In this diagram, the radius of the circle is indicated by $r$, the central angle by $\theta$ (the Greek letter theta), and the arc length by $s$, as shown.




When the central angle of a circle intercepts an arc that is the same length as the radius of the circle ($r = s$, in the circle shown), the measure of this angle ($\theta$, in the circle shown) is defined to be one radian.

<u>Vocabulary</u>:
***Radian*** – in a circle, the measure of a central angle for which the intercepted arc length equals the radius of the circle

Radians are used in almost all mathematical applications (including future high school and university–level courses) and in many computer languages; using radians will often make the work easier.

Here are some basic facts about radian measure:

A full circle is made up of $2\pi$ radians (360°).
A semicircle is made up of $\pi$ radians (180°).
A right angle is made up of $\frac{\pi}{2}$ radians (90°).

$$1° = \frac{\pi}{180} \text{ radians}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)°.$$ This is approximately 57.3°.

REMEMBER:

To denote radian measure, just write the number of radians—with no symbol. For example, to write 2 radians, just write it as 2. Be careful. If an angle measure is written with no degree symbol, it will be interpreted as radians.

The calculator has a radian mode. Be sure to know how to put your calculator in this mode.

CONVERSIONS:

Use the following formula in all conversions, plugging in all known information, then solving the proportion for whatever is unknown. Use a variable for either "radians" or for "degrees," depending on what is unknown.

$$\frac{\pi}{180} = \frac{\text{radians}}{\text{degrees}}$$

Example 1
Convert each angle expressed in degrees to radians.

a. $120^\circ$

$$\begin{aligned} \frac{\pi}{180} &= \frac{r}{120} \\ 120\pi &= 180r \\ r &= \frac{120\pi}{180} \\ r &= \frac{2\pi}{3} \end{aligned}$$

$$120^\circ = \frac{2\pi}{3}$$

b. $-245^\circ$

$$\begin{aligned} \frac{\pi}{180} &= \frac{r}{-245} \\ -245\pi &= 180r \\ r &= -\frac{245\pi}{180} \\ r &= -\frac{49\pi}{36} \end{aligned}$$

$$-245^\circ = -\frac{49\pi}{36}$$

Example 2
Convert each angle expressed in radians to degrees.

a. $\frac{\pi}{3}$

$$\begin{aligned} \frac{\pi}{180} &= \frac{\frac{\pi}{3}}{d} \\ 180\left(\frac{\pi}{3}\right) &= \pi d \\ \frac{180}{3} &= d \\ 60 &= d \end{aligned}$$

$$\frac{\pi}{3} = 60^\circ$$

b. $-\frac{3\pi}{4}$

$$\begin{aligned} \frac{\pi}{180} &= \frac{-\frac{3\pi}{4}}{d} \\ 180\left(-\frac{3\pi}{4}\right) &= \pi d \\ 180\left(-\frac{3}{4}\right) &= d \\ -135 &= d \end{aligned}$$

$$-\frac{3\pi}{4} = -135^\circ$$

Example 3
Use a calculator to evaluate.

a. $\sin 1$

With the calculator in radian mode, find the sine of 1. The display on the calculator should read about 0.84147.

b. $\sin 1^\circ$

With the calculator in degree mode, find the sine of 1. The display on the calculator should read about 0.01745.

Section A1 Exercises

Do on binder paper. Show steps.

Convert each of the following angles to radian measure. Simplify; express answers in terms of $\pi$.

1. $30^\circ$
2. $15^\circ$
3. $100^\circ$
4. $200^\circ$
5. $75^\circ$
6. $105^\circ$
7. $120^\circ$
8. $240^\circ$
9. $-320^\circ$
10. $-250^\circ$
11. $-85^\circ$
12. $-175^\circ$

Convert each of the following angles to degree measure.

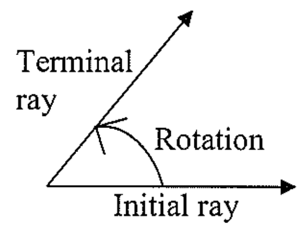
13. 1
14. 2
15. $8\pi$
16. $-12\pi$
17. $\frac{3\pi}{4}$
18. $-\frac{11\pi}{4}$

19. Find the length of an arc of a circle with a 10cm radius associated with a central angle of $\frac{2\pi}{3}$.
20. Find the length of an arc of a circle with a 5 foot radius associated with a central angle of 2.1.

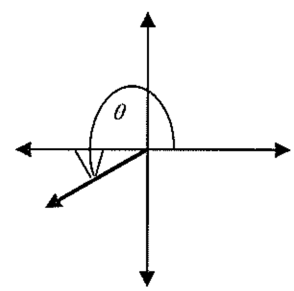
# Section A2—ANGLES IN TRIGONOMETRY

LEARNING OUTCOMES:
- Measure angles as rotations
- Determine reference angles

In Geometry, an angle is defined as two rays with a common endpoint. In trigonometry, however, an angle is a ***rotation*** from one ray to another. The rotation begins at the initial ray and ends at the terminal ray. Note the presence of arrowheads to indicate both the rays and the rotation.



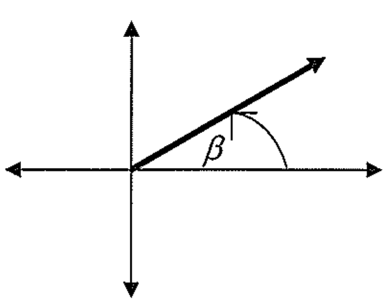

Typically, the initial ray of an angle will be the positive x–axis (unless stated otherwise). Such an angle is said to be in ***standard position***. To sketch an angle in standard position, show only the terminal ray and the rotation.

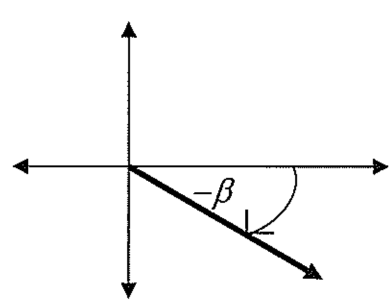



The angle shown here as $\theta$ measures $210^\circ$ and is in standard position. (Greek letters are commonly used for angles. The letter $\theta$ is called theta.)

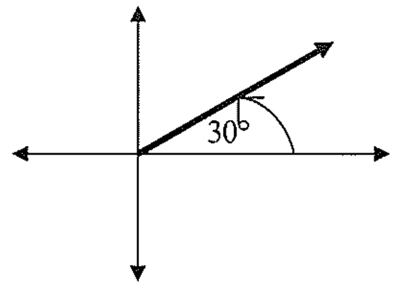
<u>*Vocabulary*</u>:
***Angle***: a rotation from one ray to another
***Initial Ray/Initial Side***: the ray at which a rotation begins
***Terminal Ray/Terminal Side***: the ray at which a rotation ends
***Standard Position***: an angle positioned such that its initial ray is on the positive x–axis
***Coterminal Angles***: angles in standard position with the same terminal ray. It is possible to find many coterminal angles if multiples of $360^\circ$ (or $2\pi$) are added to or subtracted from an angle's measure.
***Reference Angle***: the positive acute angle between the terminal ray of an angle and the $x$–axis, sometimes noted as $\theta_{ref}$.

If angles are rotations, angles measuring greater than $180^\circ$ may exist. Negative angles may exist. Counterclockwise rotations represent positive angles and clockwise rotations represent negative angles. A reference angle is the positive acute angle between the terminal ray and the nearest $x$–axis. The two angles shown are in standard position; each has a reference angle of $\beta$. Note the arrowheads indicating direction.



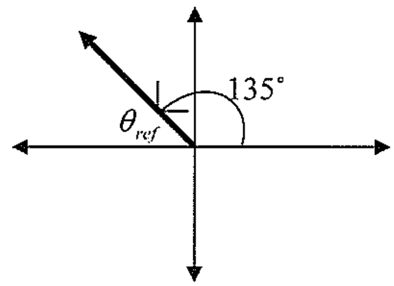
Example 1
The diagram shows a 30° angle. Name two other coterminal angles.




Solution:

One can arrive at the same terminal ray by rotating in the opposite direction. The angle whose measure is –330° has the same terminal ray. Another way is to go around more than once. One revolution is 360°, so the angle measuring (30 + 360 + 360)°, or 750°, also has the same terminal ray. Thus, 30°, –330°, and 750° are all coterminal angles. There are an infinite number of coterminal angles, as 360° may be added or subtracted any number of times.

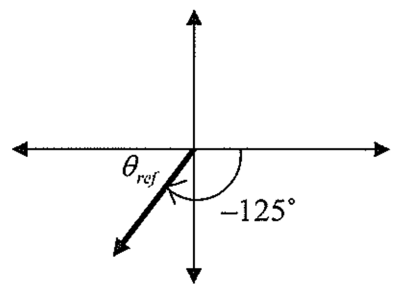
Example 2
Determine the reference angle.




Solution:

The reference angle is 180° – 135°, or 45°.

Example 3
Determine the reference angle for an angle measuring –125°.




Solution:

The reference angle is –125° – (–180°), or 55°. Note that reference angles are always positive, even when the angle measure is negative.

Section A2 Exercises

Draw the angle (show the rotation).

1. 34°
2. 230°
3. –120°
4. –175°
5. 60°
6. –400°
7. –955°
8. 855°
9. 160°
10. $-\dfrac{11\pi}{6}$
11. $8\pi$
12. $\dfrac{13\pi}{4}$

State the reference angle for each of the following angles of rotation. Note: if an angle is expressed in radians, express $m\theta_{ref}$ in radians.

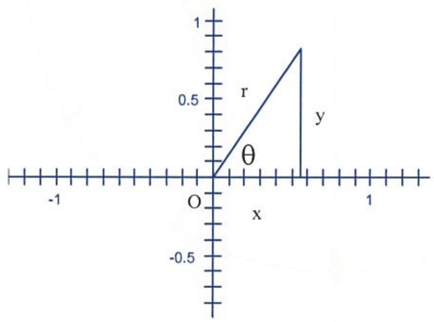
13. 405°
14. 260°
15. –600°
16. $-\dfrac{11\pi}{6}$
17. $\dfrac{2\pi}{3}$
18. $\dfrac{13\pi}{4}$

# Section A3—INTRODUCTION TO ANALYTIC TRIGONOMETRY

*Vocabulary:*
***Analytic Trigonometry*** – the study of the properties and relations of the trigonometric functions in the coordinate plane; an extension of right triangle trigonometry
***Unit Circle*** – a circle centered at the origin whose radius is one.

| |
|---|
| LEARNING OUTCOMES:<br>Determine trigonometric ratios for special angles in standard position<br>Determine the coordinates on the unit circle for terminal rays of selected rotations<br>Find exact values of trigonometric expressions without a calculator |

As defined in a previous section, the reference angle connects geometric trigonometry with new definitions in analytic trigonometry. With this diagram,




the Pythagorean Theorem becomes

$$x^2 + y^2 = r^2$$

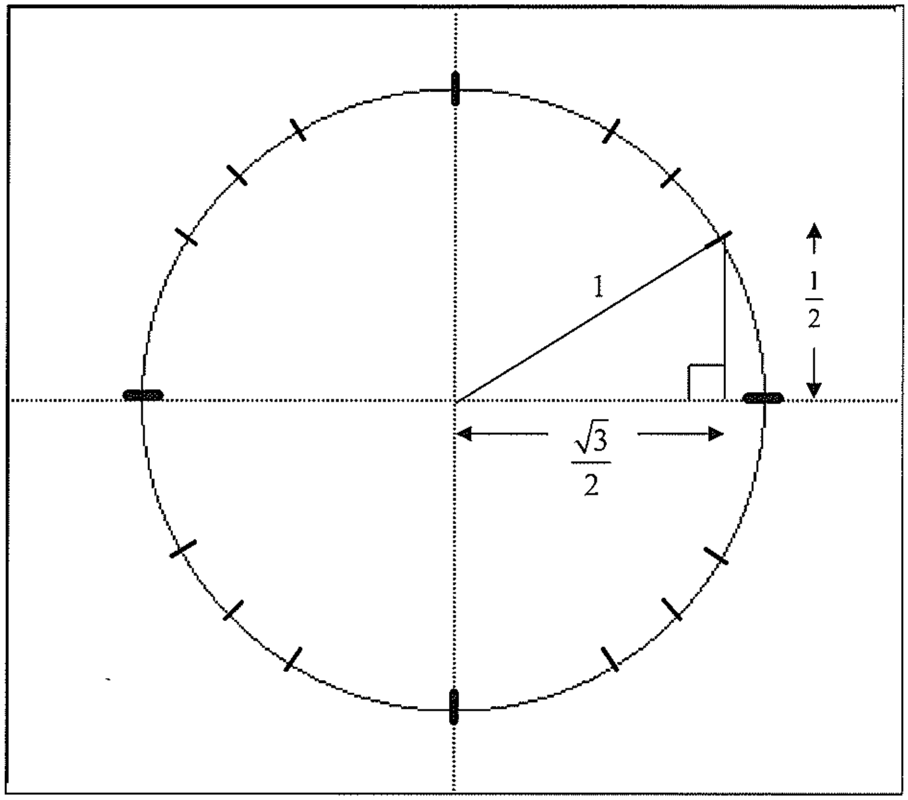
and SOHCAHTOA becomes

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

Expand on that to define three reciprocal trigonometry functions, so named because they are the reciprocals of the first three:

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Exact trigonometric values can be found for several specific angles. Recall the ratios of the sides in the special triangles—$1:1:\sqrt{2}$ in the 45°–45°–90° triangle and $1:\sqrt{3}:2$ in the 30°–60°–90° triangle. From this, coordinates for many angles in the unit circle can be determined in the unit circle. Each of the highlighted points in the four quadrants of the unit circle represent angles that have reference angles measuring either 30°, 45°, or 60°. Recall that the radius of the unit circle is 1. Analyze this diagram, which illustrates a 30° rotation:



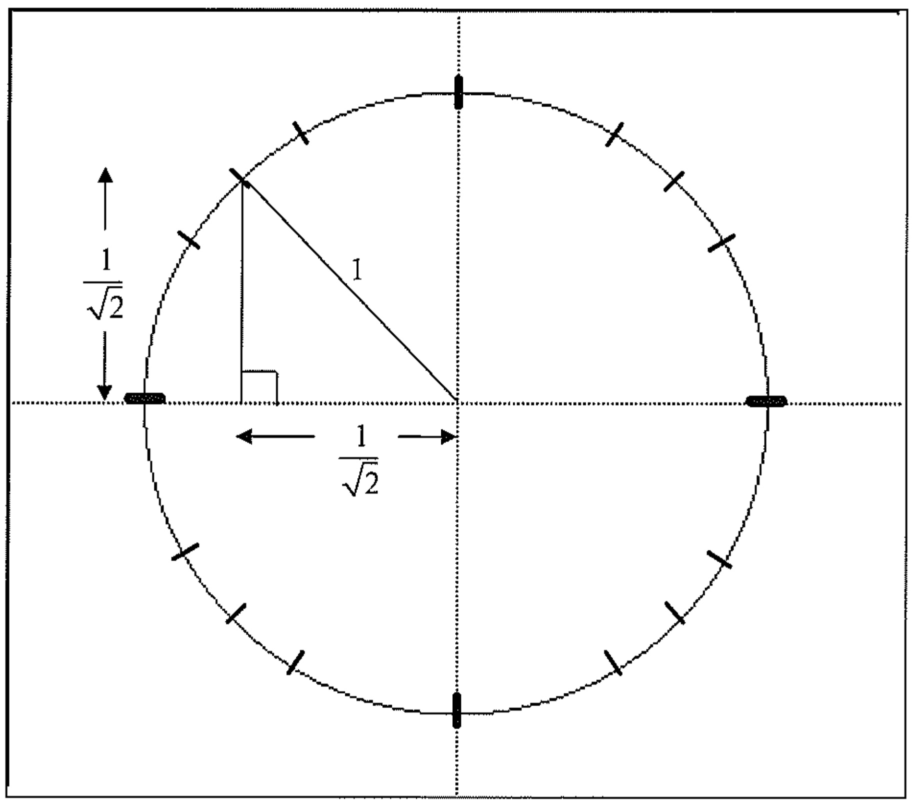

Analysis shows that the coordinates of the point at which the terminal ray of a 30° angle intersects the unit circle are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 1: Find the coordinates on the unit circle for a 135° rotation

Solution:

Sketch a 135° rotation; note that the terminal ray is in Quadrant II and $m\theta_{ref} = 45°$.

Use the $1:1:\sqrt{2}$ side ratio to find the lengths of the horizontal and vertical sides of the $45° - 45° - 90°$ triangle in the unit circle. Recall that the radius is 1.




The side lengths are all shown as positive, but in QII, the $x$–coordinate is negative, so the coordinates of the point at which the terminal ray of a 135° angle intersects the unit circle are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Continuing this process, coordinates for ALL the marked points on the unit circle can be found.

Further, exact values of trigonometric expressions may be found using the values from the unit circle in conjunction with the analytic definitions of each of the trigonometric ratios.

<u>Example 2</u>: Find the exact values of each of the following. Simplify completely.

(a) $\sin 300° \sec 300°$

$$= \frac{y}{r} \bullet \frac{r}{x} \qquad \text{(recall that } r = 1\text{)}$$
$$= -\frac{\sqrt{3}}{2} \bullet 2$$
$$= -\sqrt{3}$$

(b) $\sin^2 \frac{7\pi}{6} + \cos^2 \frac{\pi}{4}$ *** Note that $\sin^2 \theta = (\sin \theta)^2$

$$= \left(\frac{y}{r}\right)^2 \bullet \left(\frac{x}{r}\right)^2$$
$$= \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$
$$= \frac{1}{4} + \frac{1}{2}$$
$$= \frac{3}{4}$$

(c) $\sin \frac{\pi}{4} \sec \frac{7\pi}{4} + \cot \frac{5\pi}{6} \cos \frac{\pi}{3}$

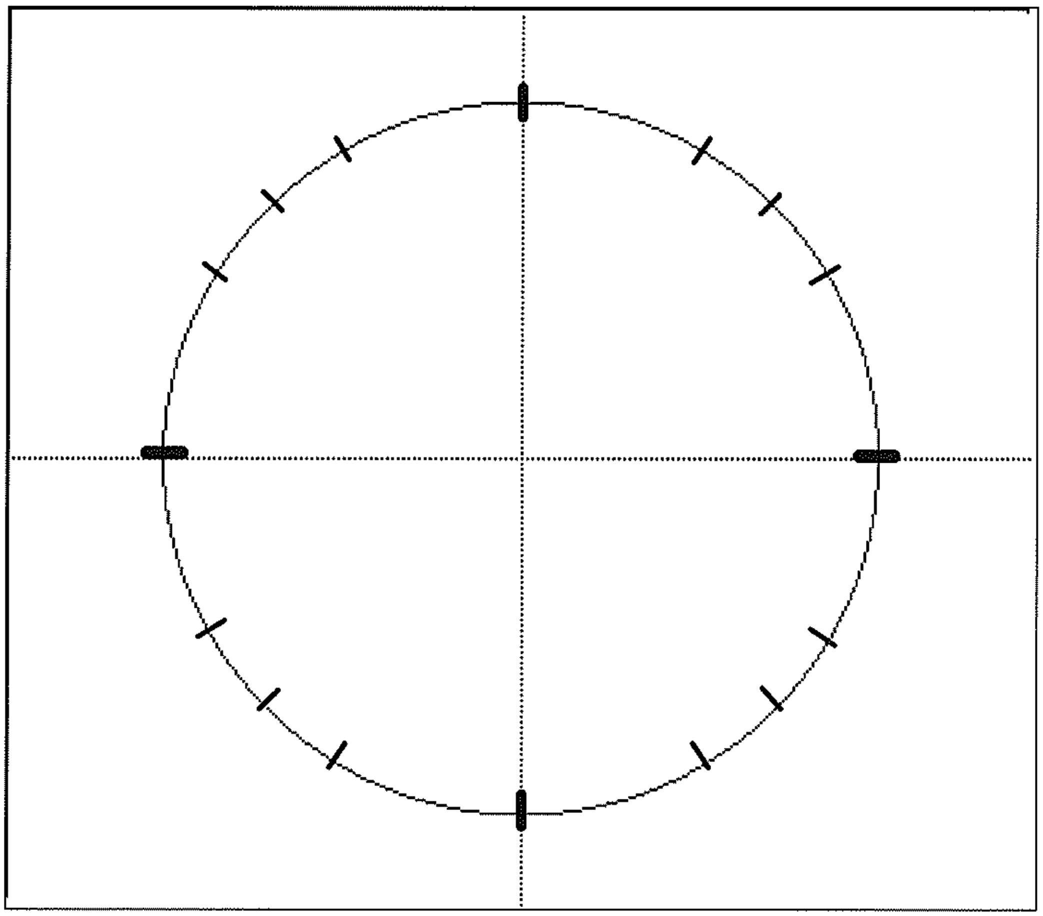
$$= \left(\frac{y}{r}\right)\left(\frac{r}{x}\right) + \left(\frac{x}{y}\right)\left(\frac{x}{r}\right)$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}\right) + \left(-\sqrt{3}\right)\left(\frac{1}{2}\right)$$
$$= 1 - \frac{\sqrt{3}}{2}$$
$$= \frac{2-\sqrt{3}}{2}$$

UNIT CIRCLE

NAME ______________________________

Date ____________________ Period ______

*A.M.D.G.*



UNIT CIRCLE—TABLE

NAME ______________________________

Date __________________ Period _____

*A.M.D.G.*

| Radians ($x$) | Degrees ( $\theta$ ) | $\cos x$ | $\sin x$ | $\tan x$ |
|---|---|---|---|---|
| | 0° | | | |
| | 30° | | | |
| | 45° | | | |
| | 60° | | | |
| | 90° | | | |
| | 120° | | | |
| | 135° | | | |
| | 150° | | | |
| | 180° | | | |
| | 210° | | | |
| | 225° | | | |
| | 240° | | | |
| | 270° | | | |
| | 300° | | | |
| | 315° | | | |
| | 330° | | | |
| | 360° | | | |

Note that $x$ in this context represents an angle, as $x$ radians.

Section A3 Exercises

1. Fill in the Unit Circle in this section with $(x, y)$ coordinates for all marked rotations.

2. Fill in the Unit Circle Table in this section.

Simplify each of the following trigonometric expressions completely **without using a calculator**.

3. $\sin^2 60° - \cos^2 60°$

4. $\sin 30° \cos 60° + \sin 60° \cos 30°$

5. $\tan^2 150° - \cot^2 210°$

6. $\sec^2 240° - \tan^2 120°$

7. $\sin^2 \dfrac{\pi}{3} + \cos^2 \dfrac{2\pi}{3}$

8. $\csc \dfrac{3\pi}{4} \tan \dfrac{\pi}{4} \cos \dfrac{7\pi}{4}$

9. $\cot \dfrac{\pi}{6} \sec \dfrac{11\pi}{6} + \csc \dfrac{7\pi}{6}$

10. $\dfrac{1}{\csc \dfrac{\pi}{4} + \cot \dfrac{5\pi}{4}} + \dfrac{1}{\csc \dfrac{3\pi}{4} - \cot \dfrac{5\pi}{4}}$

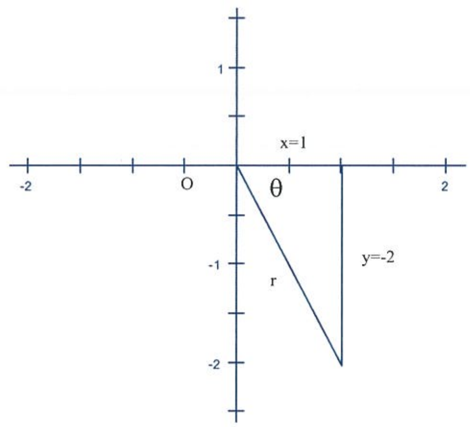
# Section A4—EXACT VALUES AND CALCULATOR USE

LEARNING OUTCOMES

- Find all six exact trigonometric ratios for an angle given a coordinate point on its terminal ray
- Use a calculator to find approximate trig values for a given angle
- Use a calculator to find all approximate angle measures for a given trig value

Previous sections have explored exact values in a unit circle. Not every circle, however, is a unit circle. How can exact values be found in a circle that is *not* a unit circle if a point of intersection on the circle is given?

Example 1: Find the six exact trig values of $\alpha$ if $(1, -2)$ is on the terminal side. In the diagram, the triangle is there for reference only, and $\theta$ represents the **reference angle**; the rotation for $\alpha$ **is not shown** because there are an infinite number of coterminal angles.




$$1^2 + (-2)^2 = r^2$$
$$r = \sqrt{5}$$

Solutions: $\sin\alpha = -\dfrac{2}{\sqrt{5}}$ $\qquad \cos\alpha = \dfrac{1}{\sqrt{5}}$ $\qquad \tan\alpha = -2$

$\csc\alpha = -\dfrac{\sqrt{5}}{2}$ $\qquad \sec\alpha = \sqrt{5}$ $\qquad \cot\alpha = -\dfrac{1}{2}$

Note that the fractions for sine and cosine are not rationalized. While some teachers—as well as the SATs—may require it, it is not emphasized here. That was done in previous classes to practice simplifying radicals.

In problems where one trig function is known, information about the quadrant is needed to tell if $x$ and $y$ are positive or negative, because the solution to the Pythagorean Theorem could be *either* (due to the need to find the square root of a number to solve for $x$ or $y$). **This is never a problem with *r*, because *r* is the <u>length</u> of the radius ray and must therefore be positive.** This will be a key issue later when inverse trig operations are explored.

Example 2: Find the other five exact trig values of $\beta$ if $\sin\beta = \frac{5}{13}$ in Quad II.

$$\begin{aligned} x^2+5^2 &= 13^2 \\ x^2 &= 144 \\ x &= \pm 12 \end{aligned}$$

in Quadrant II, $x$ is negative, so $x = -12$, $y = 5$, and $r = 13$.

Solutions: $\cos\beta = -\frac{12}{13}$ $\tan\beta = -\frac{5}{12}$

$\csc\beta = \frac{13}{5}$ $\sec\beta = -\frac{13}{12}$ $\cot\beta = -\frac{12}{5}$

Example 3: Find the other five exact trig values of $\delta$ if $\cot\delta = \frac{4}{5}$ in Quad III.

$\cot\delta = \frac{x}{y}$, but both $x$ and $y$ are negative in Q III, so

$$\begin{aligned} (-4)^2+(-5)^2 &= r^2 \\ r^2 &= 41 \\ r &= \sqrt{41} \end{aligned}$$

Solutions: $\sin\delta = -\frac{5}{\sqrt{41}}$ $\cos\delta = -\frac{4}{\sqrt{41}}$ $\tan\delta = \frac{5}{4}$

$\csc\delta = -\frac{\sqrt{41}}{5}$ $\sec\delta = -\frac{\sqrt{41}}{4}$

**Calculator Use**

Much of trigonometry will require the use of a calculator, since only certain exact values (those from the unit circle) are known. There are a few things to keep in mind with calculator problems.

REMEMBER:
1. The mode the calculator is in makes a huge difference. The user must "tell" the calculator if the angles entered are degrees or radians. Sin 3° is very different from Sin 3.
2. Calculators only have three trig keys (sin, cos, and tan), not six. The other three (csc, sec, and cot) are reciprocals of the first three.
3. Knowing a trig value alone is not sufficient to tell us the angle. Knowing one trig value only *narrows the answers down* to the reference angle in two possible quadrants.

Example 4: Find the six approximate trig values of 165°. Round all values to the nearest thousandth.

Solution:

$$\sin 165° = .259$$
$$\cos 165° = -.966$$
$$\tan 165° = -.268$$
$$\csc 165° = \frac{1}{\sin 165°} = 3.864$$
$$\sec 165° = \frac{1}{\cos 165°} = -1.035$$
$$\cot 165° = \frac{1}{\tan 165°} = -3.732$$

**Finding Angles**

When finding angles by calculator, remember that when a positive *sine* value is entered, the calculator does not "know" whether an angle in QI or QII (the two quadrants in which sine is positive) is needed. When a positive *cosine* value is entered, the calculator does not "know" if the angle is in QI or QIV. When a positive tangent value is entered, the calculator does not "know" if the angle is in QI or QIII. There are two answers to each inverse problem, but the calculator can only give one of them. In addition, there are an infinite number of coterminal angles that could represent the angles needed. The following is a way to BOTH get the alternate quadrant's answers AND list all the coterminal angles. What follows are the trig–inverse rules when angles are expressed in degrees:

$$\cos^{-1}\frac{x}{r} = \left\{\begin{matrix} \text{calculator} \pm 360°\text{n} \\ -\text{calculator} \pm 360°\text{n} \end{matrix}\right\}$$

$$\sin^{-1}\frac{y}{r} = \left\{\begin{matrix} \text{calculator} \pm 360°\text{n} \\ 180° - \text{calculator} \pm 360°\text{n} \end{matrix}\right\}$$

$$\tan^{-1}\frac{y}{x} = \left\{\begin{matrix} \text{calculator} \pm 360°\text{n} \\ 180° + \text{calculator} \pm 360°\text{n} \end{matrix}\right\} = \text{calculator} \pm 180°\text{n}$$

The answers represent *sets* of angles, so use braces to indicate the sets.

The symbols $\cos^{-1} x$, $\sin^{-1} x$ and $\tan^{-1} x$ will be used almost exclusively in this course. But, with circular functions, inverse is often replaced by "arc"—as in $\arccos x$, $\arcsin x$ and $\arctan x$.

| | |
|---|---|
| $\sin^{-1}\phi = \arcsin\phi$ | $\csc^{-1} A = \text{arc}\csc A$ |
| $\cos^{-1}\theta = \arccos\theta$ | $\sec^{-1}\beta = \text{arc}\sec\beta$ |
| $\tan^{-1} x = \arctan x$ | $\cot^{-1}\mu = \text{arc}\cot\mu$ |

Example 5: Find the following approximate angle values in degrees.

(a) Find $\alpha$ if $\sin \alpha = .351$

According to the calculator, $\sin^{-1}(.351) = 20.548°$. The rule for $\sin^{-1}$ tells us the second possible value for $\alpha$ is $180° - 20.548° = 159.452°$. Considering all coterminal angles as well,

$$\alpha = \begin{Bmatrix} 20.548° \pm 360°n \\ 159.452° \pm 360°n \end{Bmatrix}$$

(b) Find $\beta$ if $\tan \beta = 1.4$

According to the calculator, $\tan^{-1}(1.4) = 54.462°$. The rule for $\tan^{-1}$ tells us the values of $\beta$ are separated by 180°. Considering all coterminal angles as well,

$$\beta = \{54.462° \pm 180°n\}$$ This can also be written as $$\beta = \begin{Bmatrix} 54.462° \pm 360°n \\ 234.462 \pm 360°n \end{Bmatrix}$$

(c) Find $\delta$ if $\sec \delta = -1.6$

$\sec \delta = -1.6$ is really $\cos^{-1}\left(\frac{1}{-1.6}\right)$. According to the calculator, $\cos^{-1}\left(\frac{1}{-1.6}\right) = 128.682°$. The rule for $\cos^{-1}$ tells us the second possible value for $\alpha$ is $-128.682°$. Considering all coterminal angles as well,

$$\delta = \begin{Bmatrix} 128.682° \pm 360°n \\ -128.682 \pm 360°n \end{Bmatrix}$$ This can also be written as $$\delta = \{\pm 128.682° \pm 360°n\}$$

(d) Find $\varphi$ if $\csc \varphi = 0.654$

$\csc \varphi = 0.654$ is really $\sin^{-1}\left(\frac{1}{0.654}\right)$. The calculator yields an error message for this value, so

$\csc \varphi = 0.654$ DNE (Does Not Exist)

The $\sec^{-1}$ or $\csc^{-1}$ of a number whose absolute value is less than 1 does not exist (DNE), because sec and csc are defined with the hypotenuse (the longest side) as the numerator. Similarly, the $\sin^{-1}$ or $\cos^{-1}$ of a number whose absolute value is more than 1 also does not exist (DNE). Note that the angles were not found by dividing: the trigonometric inverse functions were used. While –1 as an exponent *does* mean reciprocal in other cases, **it does not mean that here.** The –1 represents the *inverse* operation, the operation that *undoes* the original. Trig–inverse is about *finding angles*. So, remember:

| | | |
|---|---|---|
| $\sin^{-1}\theta \neq \csc\theta$ | $\cos^{-1}\theta \neq \sec\theta$ | $\tan^{-1}\theta \neq \cot\theta$ |

What follows are the trig–inverse rules when angles are expressed in radians:

$$\cos^{-1}\frac{x}{r}=\begin{Bmatrix}\text{calculator}\pm 2\pi n\\ -\text{calculator}\pm 2\pi n\end{Bmatrix}$$

$$\sin^{-1}\frac{y}{r}=\begin{Bmatrix}\text{calculator}\pm 2\pi n\\ \pi-\text{calculator}\pm 2\pi n\end{Bmatrix}$$

$$\tan^{-1}\frac{y}{x}=\begin{Bmatrix}\text{calculator}\pm 2\pi n\\ \pi+\text{calculator}\pm 2\pi n\end{Bmatrix}=\text{calculator}\pm \pi n$$

Example 6: Find the following approximate angle values in radians.

(a) Find $\alpha$ if $\cos\alpha = .422$

According to the calculator, $\cos^{-1}(.422)=1.135$. The rule for $\cos^{-1}$ tells us the second possible value for $\alpha$ is $-1.135$. Considering all coterminal angles as well,

$$\alpha=\begin{Bmatrix}1.135 \pm 2\pi n\\ -1.135 \pm 2\pi n\end{Bmatrix}$$

(b) Find $\delta$ if $\csc\delta = -0.3$

$\csc\delta = -0.3$ is really $\sin^{-1}\left(\frac{1}{-0.3}\right)$. The calculator yields an error message for this value, so

$\csc\delta = -0.3$ DNE

(c) Find $\varphi$ if $\cot\varphi = 1.7$

$\cot\varphi = 1.7$ is really $\tan^{-1}\left(\frac{1}{1.7}\right)$. According to the calculator, $\tan^{-1}\left(\frac{1}{1.7}\right)=0.532$. Considering all coterminal angles as well,

$$\varphi=\{0.532 \pm \pi n\}$$

Section A4 Exercises

Find the six exact trig values if the given point is on the terminal side of $\alpha$ .

1. (6, 8)
2. (–1, –4)
3. (2, –3)
4. (–5, 5)

Find the other five exact trig values for the given value.

5. $\sin A = \frac{2}{3}$ in Quad I

6. $\cos C = -\frac{4}{9}$ in Quad III

7. $\tan B = \frac{25}{24}$ in Quad III

8. $\csc \phi = -\frac{6}{5}$ in Quad IV

9. $\sec \varphi = -\frac{\sqrt{37}}{3}$ in Quad II

10. $\cot \omega = -\frac{11}{60}$ in Quad II

Approximate the following trig values to 3 decimal places.

11. sin 14°
12. cos 5
13. tan 140°
14. cot 14
15. sec –6
16. csc –195°

Approximate the following angle values to 3 decimal places in degrees.

17. $\sin^{-1} .652$
18. arccos –.521
19. $\tan^{-1} 1.432$
20. arccot –.652
21. $\sec^{-1} 1.781$
22. $\csc^{-1} .395$

Approximate the following angle values to 3 decimal places in radians.

23. $\sin^{-1} 1.633$
24. arccos –.101
25. $\tan^{-1} 4.388$
26. arccot –.205
27. $\sec^{-1} 2.641$
28. $\csc^{-1} 1.171$

# Section A5—TRIGONOMETRIC IDENTITIES
## (Reciprocal, Quotient, and Pythagorean)

Recall:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Based on these definitions, we can create some simple identities.

**The Reciprocal Identities**

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

**The Quotient Identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\tan\theta = \frac{\sec\theta}{\csc\theta} \qquad \cot\theta = \frac{\csc\theta}{\sec\theta}$$

**The Pythagorean Identities**

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

The Pythagorean identities have alternative versions that we are likely to see as well:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$1 - \cos^2\theta = \sin^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta \qquad \csc^2\theta - 1 = \cot^2\theta$$

$$1 - \sin^2\theta = \cos^2\theta \qquad \sec^2\theta - \tan^2\theta = 1 \qquad \csc^2\theta - \cot^2\theta = 1$$

The alternative forms are useful, as they can be factored as "difference of squares" binomials. For example,

$$1 - \sin^2\theta = (1 + \sin\theta)(1 - \sin\theta)$$

| LEARNING OUTCOMES |
| --- |
| Transform basic trig expressions. <br> Prove basic trig identities. |

The proofs of the identities we will be working with are algebraic in nature. Meaning, we will show that the use of multiplication, addition and common denominators will cause one side of the equation to simplify to the other. Or, similar to what we did with proofs in Geometry, both sides simplify to the same thing.

Example 1: Prove $\csc x \tan x \cos x = 1$

$$\begin{aligned} \csc x \tan x \cos x &= 1 \\ \frac{1}{\sin x} \bullet \frac{\sin x}{\cos x} \bullet \cos x &= 1 \\ 1 &= 1 \end{aligned}$$

Notice that the answer is the process, not the final line. We knew what the final line would be, because it was given.

Example 2: Prove $\cot\beta \sec\beta + \csc\beta = 2\csc\beta$

This one can be done quickly if we see $\cot\beta = \dfrac{\csc\beta}{\sec\beta}$. This leads to

$$\begin{aligned} \cot\beta \sec\beta + \csc\beta &= 2\csc\beta \\ \frac{\csc\beta}{\sec\beta} \bullet \sec\beta + \csc\beta &= 2\csc\beta \\ \csc\beta + \csc\beta &= 2\csc\beta \\ 2\csc\beta &= 2\csc\beta \end{aligned}$$

Like proofs in Geometry, there are typically different approaches to trigonometric identity proofs. The answer is the *process*, not the *result*. In the previous example, one might not see that $\cot\beta = \dfrac{\csc\beta}{\sec\beta}$. Another, longer way would be to turn the whole problem into $\sin\beta$ and $\cos\beta$, as follows:

Example 2 (again, a different way): Prove $\cot\beta\sec\beta+\csc\beta=2\csc\beta$

$$\begin{aligned}
\cot\beta\sec\beta+\csc\beta &= 2\csc\beta\\
\frac{\cos\beta}{\sin\beta}\frac{1}{\cos\beta}+\frac{1}{\sin\beta} &= 2\csc\beta\\
\frac{1}{\sin\beta}+\frac{1}{\sin\beta} &= 2\csc\beta\\
\frac{2}{\sin\beta} &= 2\csc\beta\\
2\csc\beta &= 2\csc\beta
\end{aligned}$$

Example 3: Prove $\dfrac{1}{\csc A+\cot A}+\dfrac{1}{\csc A-\cot A}=2\csc A$

$$\begin{aligned}
\frac{1}{\csc A+\cot A}+\frac{1}{\csc A-\cot A} &= 2\csc A\\
\frac{1}{\csc A+\cot A}\left(\frac{\csc A-\cot A}{\csc A-\cot A}\right)+\left(\frac{\csc A+\cot A}{\csc A+\cot A}\right)\frac{1}{\csc A-\cot A} &= 2\csc A\\
\frac{\csc A-\cot A+\csc A+\cot A}{(\csc A+\cot A)(\csc A-\cot A)} &= 2\csc A\\
\frac{2\csc A}{\csc^2 A-\cot^2 A} &= 2\csc A\\
\frac{2\csc A}{1} &= 2\csc A\\
2\csc A &= 2\csc A
\end{aligned}$$

** Notice the Pythagorean identity in line 4 of the proof!

Techniques for Approaching Trigonometric Identity Proofs:

1. Verify that you wrote the original problem correctly before you begin.
2. Decide which side of the equation is "uglier;" that's usually the best place to start; this could be *either* the right or the left side of the equation.
3. Work vertically; keep the = signs lined up.
4. Show every step; don't assume the reader can follow your "mental leaps."
5. Use identities to reduce the problem to two or fewer trig functions. These might be sine and cosine, but be aware of how secant and tangent or cosecant and cotangent work together.
6. Squares indicate *possible* Pythagorean identities.
7. Do Algebra; remember to simplify.
    a) Common Denominators
    b) Combine "like terms"
    c) Distribute and/or FOIL
    d) Factor
8. Finish the proof by verifying that both sides of the equation *look* identical.

Notice that in Example 3, a Pythagorean identity emerged in the middle of the proof. Some examples, however, will show Pythagorean identities earlier in the proof, perhaps at the very beginning. Some of these problems will involve factoring. One of the techniques under the "Do Algebra" heading is *factor*.

LEARNING OUTCOME
Prove trigonometric identities that involve factoring.

Example 4: Prove $\sin^2 x+\cot^2 x\sin^2 x=1$

$$\begin{aligned}
\sin^2 x+\cot^2 x\sin^2 x=1 &= 1\\
\sin^2 x\left(1+\cot^2 x\right) &= 1\\
\sin^2 x\left(\csc^2 x\right) &= 1\\
\sin^2 x\left(\frac{1}{\sin^2 x}\right) &= 1\\
1 &= 1
\end{aligned}$$

Example 5: Prove $\cot^4 w-\csc^4 w=1-2\csc^2 w$

$$\begin{aligned}
\cot^4 w-\csc^4 w &= 1-2\csc^2 w\\
\left(\cot^2 w-\csc^2 w\right)\left(\cot^2 w-\csc^2 w\right) &= 1-2\csc^2 w\\
(-1)\left(\cot^2 w+\csc^2 w\right) &= 1-2\csc^2 w\\
-\left(\csc^2 w-1+\csc^2 w\right) &= 1-2\csc^2 w\\
1-2\csc^2 w &= 1-2\csc^2 w
\end{aligned}$$

Example 6: Prove $\dfrac{\cos^2\delta+2\cos\delta+1}{\cos^2\delta-3\cos\delta-4}=\dfrac{\cos\delta+1}{\cos\delta-4}$

$$\begin{aligned}
\frac{\cos^2\delta+2\cos\delta+1}{\cos^2\delta-3\cos\delta-4} &= \frac{\cos\delta+1}{\cos\delta-4}\\
\frac{(\cos\delta+1)(\cos\delta+1)}{(\cos\delta-4)(\cos\delta+1)} &= \frac{\cos\delta+1}{\cos\delta-4}\\
\frac{\cos\delta+1}{\cos\delta-4} &= \frac{\cos\delta+1}{\cos\delta-4}
\end{aligned}$$

Example 7: Prove $\dfrac{3\sec^2\theta-8\tan\theta+1}{\sec^2\theta-\tan\theta-3}=\dfrac{3\tan\theta-2}{\tan\theta+1}$

$$\begin{aligned}
\frac{3\sec^2\theta-8\tan\theta+1}{\sec^2\theta-\tan\theta-3} &= \frac{3\tan\theta-2}{\tan\theta+1}\\
\frac{3\left(\tan^2\theta+1\right)-8\tan\theta+1}{\left(\tan^2\theta+1\right)-\tan\theta-3} &= \frac{3\tan\theta-2}{\tan\theta+1}\\
\frac{3\tan^2\theta-8\tan\theta+4}{\tan^2\theta-\tan\theta-2} &= \frac{3\tan\theta-2}{\tan\theta+1}\\
\frac{(3\tan\theta-2)(\tan\theta-2)}{(\tan\theta+1)(\tan\theta-2)} &= \frac{3\tan\theta-2}{\tan\theta+1}\\
\frac{3\tan\theta-2}{\tan\theta+1} &= \frac{3\tan\theta-2}{\tan\theta+1}
\end{aligned}$$

Section A5 Exercises

Prove the following identities.

1. $\cos^2 x + \tan^2 x \cos^2 x = 1$

2. $2\cos\theta = \dfrac{\cos\theta \tan\theta + \sin\theta}{\tan\theta}$

3. $1 + 2\tan^2\beta = \sec^4\beta - \tan^4\beta$

4. $4 + (\tan\sigma - \cot\sigma)^2 = \sec^2\sigma + \csc^2\sigma$

5. $\dfrac{(\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta)}{\cos^2\theta} = \sec^2\theta$

6. $\dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\csc^2 x$

7. $-\dfrac{(-\sin\theta)}{\cos^2\theta} = \sec\theta \tan\theta$

8. $-\dfrac{\cos\varphi}{\sin^2\varphi} = -\csc\varphi \cot\varphi$

9. $\dfrac{(\cos \mathrm{a} + \sin \mathrm{a})(\cos \mathrm{a} + \sin \mathrm{a}) - 1}{2\cos \mathrm{a}} = \sin \mathrm{a}$

10. $\tan \mathrm{W}\ (\cot \mathrm{W} \cos \mathrm{W} + \sin \mathrm{W}) = \sec \mathrm{W}$

11. $\dfrac{\sec\beta}{\sin\beta} - \dfrac{\sin\beta}{\cos\beta} = \cot\beta$

12. $\dfrac{\sin\lambda}{1 - \cos\lambda} + \dfrac{1 - \cos\lambda}{\sin\lambda} = 2\csc\lambda$

13. $\cos^2\theta = \dfrac{\cot\theta \csc\theta \tan\theta - \sin\theta}{\csc\theta}$

14. $\tan \mathrm{b} + \cot \mathrm{b} = \dfrac{\csc \mathrm{b}}{\cos \mathrm{b}}$

15. $\dfrac{\cos x}{1 - \cos x} = \cot x \csc x + \cot^2 x$

16. $\dfrac{\cot\theta+1}{\cot\theta}=\dfrac{1+\tan^3\theta}{-\tan\theta+\sec^2\theta}$

17. $\dfrac{\sin^2 \mathrm{B}-\sin \mathrm{B}+1}{1-\sin \mathrm{B}}=\dfrac{1+\sin^3 \mathrm{B}}{1-\sin^2 \mathrm{B}}$

18. $\dfrac{\cot^2\alpha-4\csc\alpha-11}{\cot^2\alpha-3}=\dfrac{\csc\alpha-6}{\csc\alpha-2}$

19. $\dfrac{1+\sin\mu}{1-\sin\mu}=2\sec^2\mu+2\sec\mu\tan\mu-1$

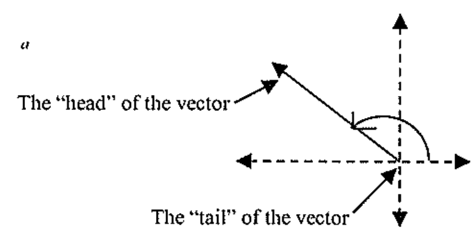
20. $\csc^6\delta-\cot^6\delta=1+3\csc^2\delta\cot^2\delta$

# Section A6—VECTORS

LEARNING OUTCOMES:
Translate a vector from polar to rectangular form and vice–versa.
Add vectors.

*Vocabulary:*
***Vector***: a directed line segment. Vector notation typically consists of a letter name for the vector with a partial arrow symbol over the name, as $\bar{a}$.
***Magnitude***: the length of a vector
***Direction***: the angle of orientation of a vector expressed in standard position
***Polar Form***: a vector expressed as a magnitude (length) and a direction (angle)
***Rectangular Form***: a vector expressed as the sum of its horizontal and vertical component parts
***Resultant Vector***: the sum of two or more vectors

Consider the following:
The vector at right, named $\bar{a}$, has magnitude 3 and direction 143°. This means that the vector is 3 units long, and its rotation is 143° from the origin. The vector is described as having a "head" and a "tail," as shown. Any vector can be described by its magnitude and direction. However, it is often useful to describe a vector in terms of its horizontal and vertical *components*; the horizontal component (indicated by an $\bar{i}$) runs in the $x$ direction, and the vertical component (indicated by a $\bar{j}$) runs in the $y$ direction.

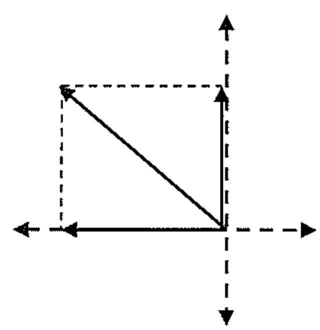



If we look at $\bar{a}$, and remember the definitions of sine and cosine,

$$\cos 143° = \frac{x}{3} \qquad \sin 143° = \frac{y}{3}$$

$$x = 3\cos 143° \qquad y = 3\sin 143°$$

$$x = -2.3959... \qquad y = 1.8054...$$



The horizontal and vertical component vectors are shown. The result of adding these two component vectors is our original vector $\bar{a}$. The horizontal and vertical parts are typically expressed as $x\bar{i}$ and $y\bar{j}$, as follows (note that the $x$ and $y$ values effectively become *coefficients* of the $\bar{i}$ and $\bar{j}$ vectors):

$$\bar{a} = -2.396\bar{i} + 1.805\bar{j}$$

When two vectors are added, each is typically expressed as the sum of its components and like terms (all the $\bar{i}$ terms and all the $\bar{j}$ terms) are added. The sum vector is typically called the resultant vector and labeled $\bar{r}$.

Example 1: If $\bar{a}$ has a magnitude of 5 and direction 70°, and $\bar{b}$ has a magnitude of 6 and direction 25°, find the resultant vector, $\bar{r}$, as the sum of two components.

Solution:

$$\bar{a} = 5\cos 70°\bar{i} + 5\sin 70°\bar{j}$$
$$\bar{b} = 6\cos 25°\bar{i} + 6\sin 25°\bar{j}$$

Note that both $\bar{i}$ vectors are horizontal, and both $\bar{j}$ vectors are vertical. Therefore, the $\bar{i}$ terms are like terms, as are the $\bar{j}$ terms. By combining like terms,

$$\left(\bar{a} + \bar{b}\right) = \bar{r} = 7.148\bar{i} + 7.234\bar{j}$$

It is sometimes either necessary or desirable to express a resultant vector not as a sum of its vertical and horizontal components, but as a magnitude (length) and direction (angle measured either in degrees or radians). The magnitude is determined using the horizontal ($\bar{i}$) and vertical ($\bar{j}$) coefficients and the Pythagorean Theorem. The direction is found using inverse trigonometry; typically $\cos^{-1}$ is used.

Example 2: Express your answer to the example 1 (above) as a magnitude and direction.

Solution:

$$x^2 + y^2 = r^2$$
$$(7.148)^2 + (7.234)^2 = r^2$$
$$r = \sqrt{(7.148)^2 + (7.234)^2}$$

$r = 10.170$ This represents the magnitude of the resultant vector $\bar{r}$

$\theta = \cos^{-1}\left(\frac{I}{R}\right)$. Use stored values.

$\theta = \pm 45.344°$

Decide whether the angle should be positive or negative based on the quadrant of the resultant (look at the signs of the $x$ and $y$ values above). This angle is in the 1st quadrant, so the angle is positive.

Therefore, the magnitude and direction are: 10.170 units at 45.344°

A common application of vectors is *velocity*, in which the speed expresses the magnitude and the bearing represents the direction of the vector.

Example 3: A plane flies at 200 mph along a bearing of 320°. The air is moving with a wind speed of 60 mph along a bearing of 190°. Find the plane's resultant velocity (speed and bearing) by adding these two velocity vectors.

Solution:

The speed represents the magnitude of each vector, and the bearing represents the direction. Solve by adding the horizontal and vertical component vectors. Then use the Pythagorean Theorem and $\cos^{-1}$ to translate the answer to speed (magnitude) and bearing (direction):

$$\begin{array}{l} 200\cos 320°\vec{i} + 200\sin 320°\vec{j} \\ 60\cos 190°\vec{i} + 60\sin 190°\vec{j} \end{array} = 94.120\vec{i} - 138.976\vec{j}$$

Speed = $\sqrt{I^2 + J^2} \approx 167.848$ mph (note that this is the length of resultant vector $\vec{r}$)

Direction = $\cos^{-1}\left(\dfrac{I}{R}\right) \approx \pm 55.893°$

The $\vec{i}$ and $\vec{j}$ coefficients above tell us that the resultant vector's angle is in QIV, so the plane's resultant velocity is 167.848 mph at –55.893°. Note that 167.848 mph at 304.107° is also an acceptable answer, because –55.893° and 304.107° are coterminal angles.

Section A6 Exercises

For problems 1–4:

- Part I: Add $\vec{a}$ to $\vec{b}$ and find the resultant vector, $\vec{r}$, as the sum of horizontal and vertical components. Round to the nearest thousandth.
- Part II: Express your answer as a magnitude and direction. Use stored values.

1. $\vec{a} = 21$ units at $\theta = 70°$
   $\vec{b} = 40$ units at $\theta = 120°$
2. $\vec{a} = 12$ units at $\theta = 160°$
   $\vec{b} = 8$ units at $\theta = 310°$
3. $\vec{a} = 15.7$ units at $\theta = 113°$
   $\vec{b} = 854.2$ units at $\theta = 21°$
4. $\vec{a} = 9.6$ units at $\theta = 199°$
   $\vec{b} = 18$ units at $\theta = 12°$
5. A plane flies at 450 mph on a bearing of 215°. The wind speed is 70 mph, and is blowing on a bearing of 80°. Find the plane's resultant velocity.
6. A cruise ship sails at 24 knots (knots = nautical miles per hour) on a bearing of 180°. The current is flowing at 4 knots on a bearing of 135°. Find the cruise ship's resultant velocity.

# Section A7—LAW OF COSINES

LEARNING OUTCOME:
Use the Law of Cosines to solve an oblique triangle.

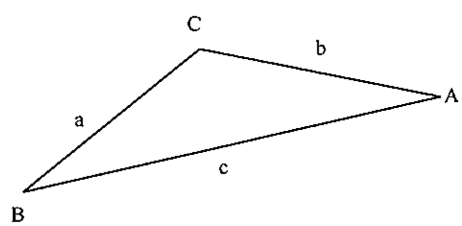
Recall that a vector is a directed line segment with two components: **magnitude** and **direction**. The magnitude of a vector is also known as its length, while the direction of a vector is an indication of the angle of orientation of the vector if it is in standard position.

In the real world, vectors are typically not in standard position, nor are they typically sketched on a coordinate plane. Under these conditions, it may be more challenging to add two vectors together. We need a new tool, called the ***Law of Cosines***. The Law of Cosines may be used to solve **any** oblique triangle, unlike SOHCAHTOA, which can only be used for right triangles.

**The Law of Cosines**: In any $\Delta ABC$,

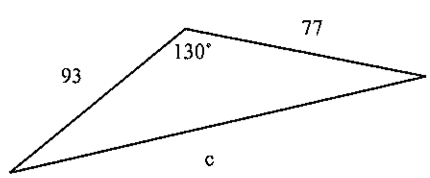
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Angles are indicated with capital letters, while the side lengths opposite those angles are indicated with matching lower case letters. The Law of Cosines can be used to solve for either *sides* or *angles*.




Example 1

Solve for c.

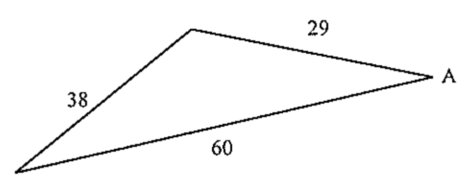



Solution:

$$c^2 = 93^2 + 77^2 - 2(93)(77)\cos 130°$$
$$c = \sqrt{93^2 + 77^2 - 2(93)(77)\cos 130°}$$
$$c \approx 154.221$$

Example 2

Solve for $m\angle A$.

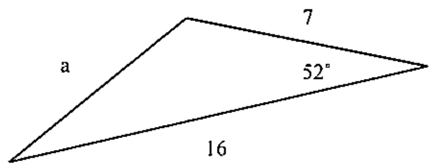



Solution:

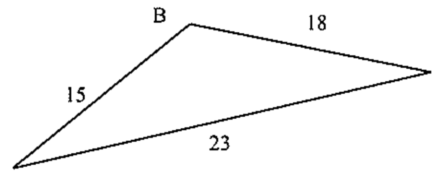
$$38^2 = 29^2 + 60^2 - 2(29)(60)\cos A$$
$$\cos A = \frac{38^2 - 29^2 - 60^2}{-2(29)(60)}$$
$$m\angle A = \cos^{-1}\left(\frac{38^2 - 29^2 - 60^2}{-2(29)(60)}\right)$$
$$m\angle A \approx 30.548°$$

Guided Practice. Solve as indicated:

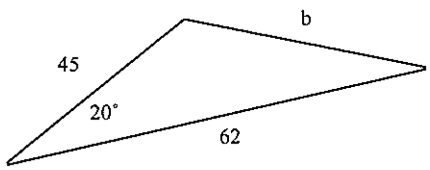
Solve for a.



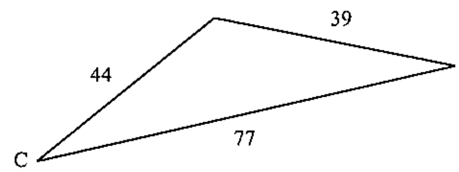

Solve for $m\angle B$.




Solve for b.




Solve for $m\angle C$.



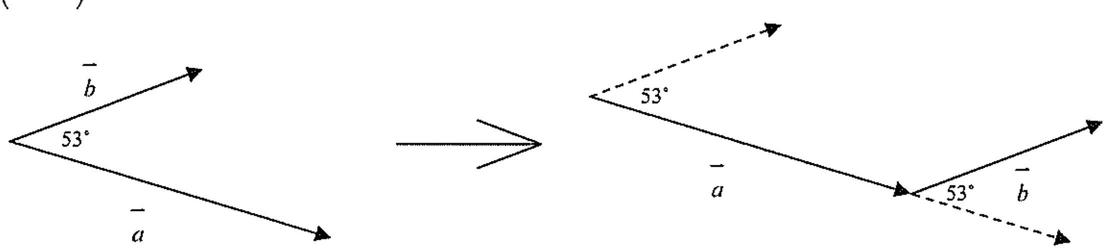

This solution technique may be applied to vectors:

<u>Example 3</u>

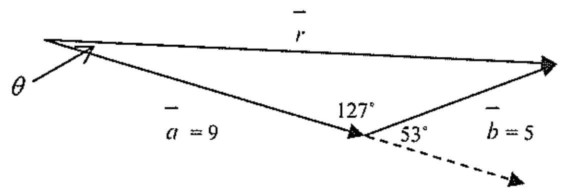
Vector $\vec{a}$ has a magnitude of 9 and vector $\vec{b}$ has a magnitude of 5. The angle between the tails of the vectors is 53°. Find the magnitude of the resultant vector $\vec{r}$ and the angle the tail of $\vec{r}$ makes with the tail of $\vec{a}$.

Solution:

Vectors $\vec{a}$ and $\vec{b}$ are oriented tail–to–tail, but in order to add vectors, they must be placed head–to–tail. To accomplish this, a vector parallel to $\vec{b}$ will be placed at the head of $\vec{a}$, as shown. The resultant vector $\vec{r}$ [also known as $(\vec{a}+\vec{b})$] will go from the tail of $\vec{a}$ to the head of the translated $\vec{b}$.



Use the new diagram (below) and solve for the length of $\bar{r}$ using the Law of Cosines. Note the location of both $\bar{r}$ and $\theta$.




$$\bar{r}^2 = 9^2 + 5^2 - 2(9)(5)\cos 127°$$
$$\bar{r} = \sqrt{9^2 + 5^2 - 2(9)(5)\cos 127°}$$
$$\bar{r} \approx 12.656$$

The angle that $\bar{a}$ makes with $\bar{r}$ (when they are considered tail–to–tail) can also be found using the Law of Cosines. Use the stored value (from your calculator) of $\bar{r}$ in your calculation rather than the rounded value.

$$5^2 = 9^2 + 12.656^2 - 2(9)(12.656)\cos\theta$$
$$\cos\theta = \frac{5^2 - 9^2 - 12.656^2}{-2(9)(12.656)}$$
$$\theta = \cos^{-1}\left(\frac{5^2 - 9^2 - 12.656^2}{-2(9)(12.656)}\right)$$
$$\theta \approx 18.393°$$

Section A7 Exercises

1. In $\Delta PRQ$, $PR = 60$, $RQ = 80$, and $m\angle R = 47°$. Find $PQ$.
2. In $\Delta WXY$, $WX = 22$, $XY = 41$, and $m\angle X = 112°$. Find $WY$.
3. Find the measures of all angles in $\Delta ABC$ if $AB = 6$, $BC = 12$, and $AC = 8$.
4. Solve for the measure of the smallest angle in a triangle whose side lengths are 7, 18, and 22.

Given vectors $\bar{a}$ and $\bar{b}$ and the angle between the vectors' tails, find $\bar{r}$ [AKA $(\bar{a}+\bar{b})$] and the angle the tail of $\bar{r}$ makes with the tail of $\bar{a}$.

5. $\bar{a} = 14$, $\bar{b} = 22$, angle between the tails of $\bar{a}$ and $\bar{b} = 119°$
6. $\bar{a} = 10$, $\bar{b} = 30$, angle between the tails of $\bar{a}$ and $\bar{b} = 122°$
7. $\bar{a} = 7$, $\bar{b} = 11$, angle between the tails of $\bar{a}$ and $\bar{b} = 73°$
8. $\bar{a} = 19$, $\bar{b} = 28$, angle between the tails of $\bar{a}$ and $\bar{b} = 12°$

# Section A8—LAW OF SINES

The Law of Sines serves as the oblique version of the SOH part of SOHCAHTOA. It relates the angles to the sides in a triangle. SOH is the specific case of the Law of Sines when the angle = 90°.

**Law of Sines**: In any $\Delta ABC$,

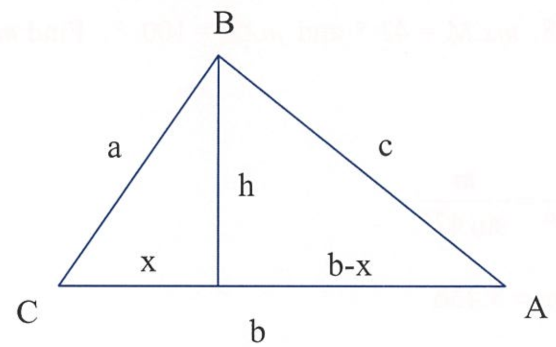
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(either form is acceptable; be consistent)

Proof of the Law of Sines, where $\Delta ABC$ is a right triangle whose right angle is $\angle B$ and whose altitude is $h$. Therefore, $h = a \sin C$.




Since the area of a triangle is Area $= \frac{1}{2}$ bh , in this case

$$\textbf{Area} = \frac{1}{2} ab \sin C$$

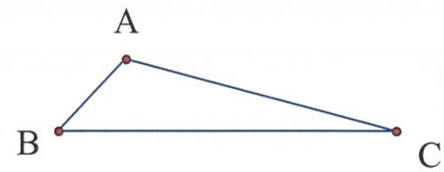
Note that we could have drawn the altitude from A or B, which would have yielded Area $= \frac{1}{2} bc \sin A$ and Area $= \frac{1}{2} ac \sin B$ respectively. Since all three must give the same area, then

Dividing by $\frac{1}{2} abc$, we get $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, which is the Law of Sines.

| LEARNING OUTCOME |
|---|
| Use the Law of Sines to find missing sides in an oblique triangle. |

The best time to use the Law of Sines is when given information about two angles and a side.

Example 1: In $\Delta ABC$, a = 6, $m\angle$B = 50 ° and $m\angle$C = 13 °. Find *b* and *c*.




Solution:

Given two angles, we know the third ($m\angle A = 117°$) because the three angles of a triangle add to 180°.

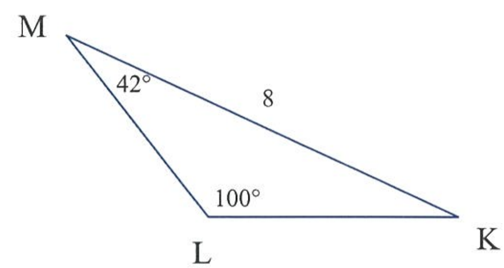
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{6}{\sin 117°} = \frac{b}{\sin 50°} \quad \text{and} \quad \frac{6}{\sin 117°} = \frac{c}{\sin 13°}$$

$$b = 5.159 \qquad\qquad c = 1.515$$

Example 2: In $\Delta MLK$, $l = 8$, $m\angle$M = 42 ° and $m\angle$L = 100 °. Find *m* and *k*.

Solution:




$$\frac{8}{\sin 100°} = \frac{m}{\sin 42°}$$

$$m = 5.436$$

Because the three angles of a triangle have a sum of 180°, $m\angle K = 38°$.

$$\frac{8}{\sin 100°} = \frac{k}{\sin 38°}$$

$$k = 5.001$$

Example 3: Find the areas of $\Delta ABC$ and $\Delta MLK$ from Examples 1 and 2.

Solution:

$\Delta ABC$: Area $= \frac{1}{2}ab\sin C = \frac{1}{2}(6)(5.159)\sin 13° = 3.482$ square units

$\Delta MLK$: Area $= \frac{1}{2}(lm)\sin K = \frac{1}{2}(8)(5.436)\sin 38° = 13.387$ square units

Section A8 Exercises

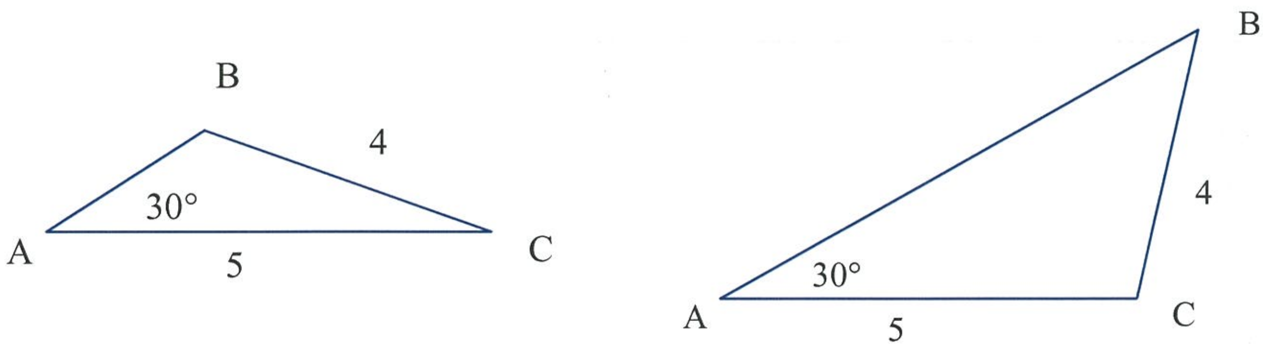
Find the missing sides and the areas of these triangles.

1. In $\Delta ABC$, $m\angle C = 35°$ , $m\angle A = 46°$ , and $b = 7$
2. In $\Delta DEF$ , $m\angle D = 35°$ , $m\angle E = 47°$ , and $f = 8$
3. In $\Delta MLK$ , $m\angle K = 63°$ , $m = 5$ and $m\angle L = 61°$
4. In $\Delta PDQ$, $m\angle P = 114°$ , $m\angle D = 17°$ , and $q = 45$
5. In $\Delta BFC$, $m\angle B = 135°$, $c = 6$ and $m\angle C = 29°$
6. In $\Delta RCQ$, $m\angle R = 33°$, $m\angle Q = 57°$, and $q = 11$.

# Section A9—AMBIGUOUS TRIANGLES

SSA--The Ambiguous Case

In Geometry, there is no SSA congruence theorem, because there might be two different ways to draw a figure with two given sides and a given non-included angle. Consider $\Delta ABC$ where $a = 4$, $b = 5$ and $m\angle A = 30°$. Both these pictures fit the criteria. This is an "ambiguous" situation because it can be understood in two different ways. If solving for *c*, there would be two answers. There are even two ways to solve for *c*.




| LEARNING OUTCOME |
|---|
| Use a variety of techniques to analyze ambiguous oblique triangles and find all possible solutions. |

<u>Example 1:</u> In $\Delta ABC$, $a = 4$, $b = 5$ and $m\angle A = 30°$. Find *c*.

Solution:

Use the Law of Cosines; be sure the given angle and side are in the correct positions in the equation.

$$a^2 + b^2 - 2ab\cos C = c^2$$
$$5^2 + c^2 - 2(5)(c)\cos 30° = 4^2$$
$$c^2 - 8.660c + 9 = 0$$

Use the Quadratic Formula to find *c*.

$$c = \frac{8.660 \pm \sqrt{8.660^2 - 4(1)(9)}}{2}$$

$$c = 7.453 \text{ or } 1.208$$

Example 1 again: In $\Delta ABC$, $a = 4$, $b = 5$ and $m\angle A = 30°$. Find $c$.

Use the Law of Sines to find the other angles. Here, the ambiguity arises from the fact that there are two angles that have any given sine value: an acute angle and an obtuse angle. The second angle—which the calculator does not give you directly—is the supplement of the first angle.

$$\frac{\sin B}{5} = \frac{\sin 30°}{4}$$
$$\sin B = \frac{5\sin 30°}{4} = 0.625$$
$$m\angle B = \sin^{-1} .625 = \begin{Bmatrix} 38.682° \\ 141.318° \end{Bmatrix}$$

The two answers for $m\angle B$ yield two answers for $m\angle C$, based on triangle sum.

$$m\angle C = \begin{Bmatrix} 111.318° \\ 8.682° \end{Bmatrix}$$

$$\frac{4}{\sin 30°} = \frac{c}{\sin 111.318°} \qquad \frac{4}{\sin 30°} = \frac{c}{\sin 8.682°}$$

$$c = 7.453 \qquad c = 1.208$$

If that is not ambiguous enough, the SSA situation is still more so. There might not be two triangles. There might only be one—or even none.

Example 2: In $\Delta ELP$, $e = 4$, $l = 5$ and $m\angle L = 34°$. Find $p$.

Solution:

$$\frac{\sin E}{4} = \frac{\sin 34°}{5}$$
$$\sin E = \frac{4\sin 34°}{5} = 0.447$$
$$m\angle E = \sin^{-1} .447 = \begin{Bmatrix} 26.574° \\ 153.426° \end{Bmatrix}$$

If $m\angle E = 26.574°$, then $m\angle P = 119.426°$. But if $m\angle E = 153.426°$, $m\angle P$ would have to be negative to get a sum of $180°$ in the triangle. Therefore,

$$m\angle P = 119.426°$$

$$\frac{5}{\sin 34°} = \frac{p}{\sin 119.426°}$$

$$p = 7.788$$

If the Law of Cosines had been used, one of the sides would have been negative.

Example 3: In $\Delta BTW$ , $b = 2$, $t = 5$ and $m\angle B = 63\ °$. Find $w$.

Solution:

$$\frac{\sin T}{5} = \frac{\sin 63°}{2}$$
$$\sin T = \frac{5\sin 63°}{2} = 2.228$$
$$m\angle T = \sin^{-1} 2.228 = \text{error}$$

There is no such triangle because side $b$ is too short to reach side $w$ with the given $m\angle B$.

## Section A9 Exercises

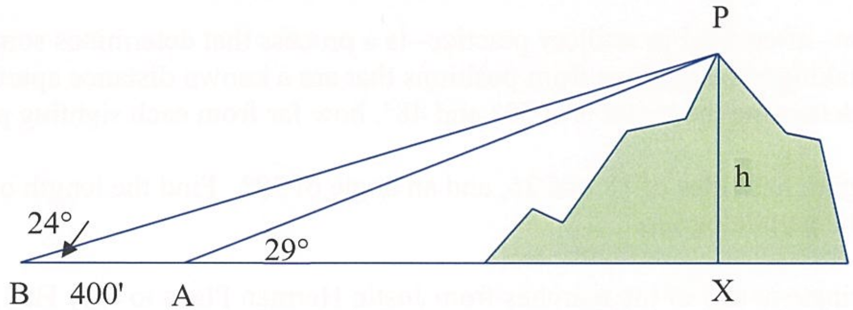
Find all possible missing side lengths and angle measures in the given triangles.

1. In $\Delta KRL$, $k = 8$, $r = 15$ and $m\angle K = 31°$
2. In $\Delta DEF$ , $m\angle D = 35°$, $d = 11$ and $f = 8$
3. In $\Delta MLK$ , $m\angle K = 163°$, $m = 5$ and $k = 8$
4. In $\Delta PDQ$, $m\angle P = 64°$, $d = 17$ and $p = 6$
5. In $\Delta BFC$, $m\angle B = 35°$, $c = 6$ and $b = 5$
6. In $\Delta RCQ$, $m\angle R = 30°$, $r = 5$ and $q = 10$
7. In $\Delta JAI$ , $j = 9$, $a = 17$ and $i = 11$
8. In $\Delta KRL$, $k = 8$, $r = 15$ and $m\angle L = 76°$

# Section A10—MODELING WITH TRIANGLES

| LEARNING OUTCOME |
| --- |
| Solve mathematical problems involving triangles. |

Example 1: Finding the height of a mountain is not as easy as finding the height of a building because you cannot directly measure the distance to a spot directly below the peak. Instead, measure the angle to the peak from two different positions. At point A, the angle of elevation to the peak is 29°. At point B, 400' further away, the angle of elevation is 24°. How tall is the mountain?




Solution:

$\Delta APX$ is a right triangle. If we knew $AX$ or $AP$, we could use SOHCAHTOA to find $PX$ (the height $h$). $AP$ is a side in $\Delta ABP$; since we have two angles in that triangle, we use the Law of Sines to find $AP$.

$m\angle XAP = 29°$, therefore $m\angle BAP = 151°$.
$m\angle B = 24°$, so $m\angle BPA = 5°$.

By the Law of Sines,

$$\frac{AP}{\sin 151°} = \frac{400}{\sin 5°}$$

$$AP = \frac{400 \sin 151°}{\sin 5°} = 2225.027'$$

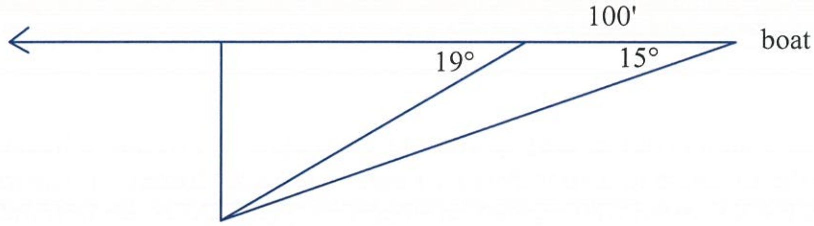
And

$$\sin 24° = \frac{h}{2225.027}$$

$$h = 2225.027 \sin\ 24° = 905'$$

Section A10 Exercises

1. A boat is sailing west toward the Golden Gate Bridge. At one time, the angle measured from due west to Fort Point is 15°. After sailing 100 feet, the angle is 19°; then the boat continues sailing along its path. How close will the boat come to Fort Point?




2. Triangulation--often used in artillery practice--is a process that determines someone's (or something's) position by taking two sightings from positions that are a known distance apart. If sights that are 200 yards apart determine the target is at 53° and 48°, how far from each sighting position is the target?

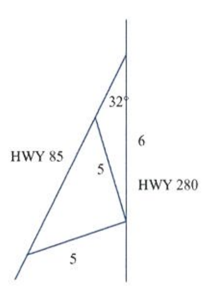
3. A parallelogram has sides of 18 and 26, and an angle of 39°. Find the length of the longer diagonal and the area of the parallelogram.

4. If you participate in any of the marches from Justin Herman Plaza to City Hall, you walk 1.5 miles along Market street, then a quarter mile on Grove. Grove meets Market at angle of 140°. How far, in a straight line, is your starting point from the end?

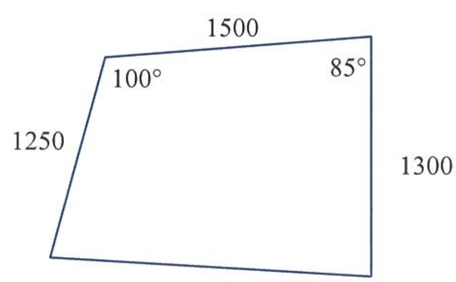
5. A swimmer sees two alligators. He tells you that the distance to one alligator is 30', the distance between the alligators is 20' and the angle where the swimmer is measures 58°.
   a. Show that he is wrong about the measure of the angle.
   b. Find the two possible distances between the swimmer and the second alligator using the correct angle, 28°.

6. A surveyor measures three sides of a triangular field and finds them to be 102', 176' and 247'. What is the area of the field?

7. A trucker on HWY 280 has a CB radio with a range of 5 miles. HWY 280 intersects HWY 85 at a 32° angle. When the trucker is 6 miles south of the intersection, between what two distances from the intersection can another trucker on HWY 85 receive the signal?




Problem 7




Problem 8

8. The surface of the Sunset Reservoir at 24th and Quintara is a quadrilateral with sides 1500', 1250', and 1300' and angles 100° and 85°. Find the other sides and angles and the surface area of the Reservoir.

# Unit B:

# Selected Topics

# Section B1—REVIEW OF LOGARITHMIC AND EXPONENTIAL EQUATIONS

| LEARNING OUTCOME |
| --- |
| Simplify and solve equations involving exponential and/or logarithmic functions. |

*Vocabulary*

***Logarithmic Function*** – $y = \log_a x$ if and only if $a^y = x$ for $a > 0$ and $a \neq 1$

***Common Log*** – base 10

***Natural Log*** – base $e$

***e*** – an irrational number, whose value is approximately 2.71828…(can you find $e$ on your calculator?)

Any logarithmic equation can be written using either "exponential form," or "logarithmic form," based on the definition of a logarithmic function shown above.

| **Common Log:** | **Natural Log:** |
| --- | --- |
| $\log_{10} N = k$ or just $\log N = k$ | $\ln_e N = k$ or just $\ln N = k$ |
| In exponential Form, $10^k = N$ | In exponential Form, $e^k = N$ |

There are three parts to any logarithmic or exponential equation: the base, the exponent, and the argument. In general,

$$\log_{base} \text{argument} = \text{exponent}$$
$$base^{exponent} = \text{argument}$$

Guided Practice. Complete each of the following examples.

Example 1: Write the equation in exponential form.

$log_3 81 = 4$

Example 2: Write each in logarithmic form.

(a) $5^3 = 125$ (b) $9^{3/2} = 27$ (c) $16^{-3/4} = \frac{1}{8}$

There are 3 types of exponential equations. The variable can be either in the base, the exponent or the argument of the equation, like these:

$x^3 = 125$ $\qquad$ $9^x = 27$ $\qquad$ $16^{-3/4} = x$

To solve an equation with a log in it, translate the equation to the more useful exponential form before you solve. Solving the equation may then require the use of a logarithm.

<u>Example 3:</u> Solve for $x$ algebraically.

(a) $\log_5 625 = x$

(b) $\log_2 \dfrac{1}{8} = x$

(c) $\log_x 4 = \dfrac{1}{8}$

(d) $\log_{\frac{1}{3}} x = \sqrt[3]{81}$

(e) $\log_3 (x+1) = 2$

(f) $\log_x 4 = -2$

For more complicated equations or translations, use the rules for logs and exponents (as follows). Note the similarities.

| *LAWS OF EXPONENTS* | *LAWS OF LOGS* |
|---|---|
| Products: $a^p \bullet a^q = a^{p+q}$ | Products: $\log_a MN = \log_a M + \log_a N$ |
| Quotients: $\dfrac{a^p}{a^q} = a^{p-q}$ | Quotients: $\log_a \dfrac{M}{N} = \log_a M - \log_a N$ |
| Power of a Power: $\left(a^p\right)^q = a^{pq}$ | Power of a Power: $\log_a M^n = n\log_a M$ |

Example 4: Use the Laws of Logs to expand the log.

$$\log\left(\frac{x^{-\frac{1}{2}}y^3}{w^0 z^{\frac{4}{5}}}\right)$$

Example 5: Use the Laws of Logs to express as a single logarithm of a single argument.

(a) $\frac{1}{3}\log_4 64 - 4\log_4 2$ (b) $1 - 3\log_5 x$

Example 6: Solve for $x$.

(a) $2\log_3 6 + \log_3 4 = x$ (b) $\ln(2x+1) - \ln(x-3) = 2\ln 3$

Example 7: Solve for x.

(a) $\log_2\left(x^2-9\right)=4$

(b) $\log_a\left(3x+5\right)+\log_a\left(x-5\right)=\log_a 7$

(c) $\frac{1}{2}\log_5\left(x+6\right)-\log_5 x=0$

(d) $2\log_3 x-\log_3\left(x-2\right)=2$

These problems can also be solved by graphing calculator. Your calculator is programmed in Base 10 or Base $e$ mode ONLY! We have to use the following rule to change the base to Base 10 or Base $e$.

**Change of Base Rules:**

$$\log_a x=\frac{\log_{10} x}{\log_{10} a} \quad or \quad =\frac{\ln x}{\ln a}$$

$$a^x=e^{x\cdot\ln a}$$

Example 8: Use your calculator to solve for $x$

(a) $\frac{4^{2-t}}{3}=7$

(b) $10^{\left(x^2-1\right)}=1000$

(c) $3^{(x-1)}=7^{(x+2)}$

Section B1 Exercises

Simplify these logarithmic expressions to a single log of a single argument.

1. $\frac{1}{3}\ln 27 + 2\ln 2$

2. $\frac{1}{3}\left(\log 52 + 2\log 4 - \log 13\right)$

3. $\frac{1}{2}\log_5 16 - 2\log_5 10$

Expand these expressions to the sum or difference of logarithms.

4. $\log\left(\dfrac{x^2 w^{-\frac{1}{3}}}{y^{\frac{1}{2}} z^{-3}}\right)$

5. $\log\left(\dfrac{x^3 w^{-\frac{2}{3}} y^{\frac{2}{5}}}{z^4}\right)$

6. $\log\left(\dfrac{x^{-\frac{1}{3}} y^4}{w^{-3} z^{\frac{2}{5}}}\right)$

Solve for $x$.

7. $\log_{\sqrt{3}}\left(\dfrac{1}{9}\right) = x$

8. $\log_3\left(\dfrac{1}{729}\right) = x$

9. $\log_4(x-1) + \log_4(3x+1) = 3$

10. $\log_6 x + \log_6(x+5) = 2$

11. $\frac{1}{2}\log_7 x = \log_7 20 - 2\left(\log_7 2 + \log_7 5\right)$

12. $\log_{12}\left(x^2+1\right) - \log_{12}(x-1) = 3$

13. $17\left(10^x\right) = 51$

14. $25^{\left(x^2-1\right)} = 125^{(x-1)}$

15. $2^{(3x-1)} = 8^{(1-x)}$

16. $7^{\left(x^2-1\right)} = 5^{(x+2)}$

# Section B2—MODELING WITH LOGARITHMIC AND EXPONENTIAL FUNCTIONS

LEARNING OUTCOME
Solve real-world problems involving Exponential and Logarithmic operations.

***TYPE I: GROWTH & DECAY***

***Population Doubling Growth*** $y = A(2)^{\frac{t}{k}}$

***Radioactive Half–Life*** $y = A\left(\frac{1}{2}\right)^{\frac{t}{k}}$

$y$ = ending amount, $A$ = initial amount, $t$ = duration of time,
$k$ = half–life or doubling time

Example 1: ***BACTERIA POPULATION PROBLEM*** If there are 100 bacteria in a petri dish and the number of bacteria doubles every 10 minutes, how long will it take for there to be a million bacteria?

Example 2: ***HALF LIFE PROBLEM*** Radium has a half–life of 1690 years. How much of a 75 g sample will be left in 400 years?

***TYPE II: FINANCES (COMPOUND INTEREST)***

*Vocabulary*:
***Simple Interest*** – interest earned on principal
***Compound Interest*** – interest calculated on the initial principal and also on the accumulated interest of previous periods of a deposit
***Annuity*** – a fixed sum of money saved or paid each period on a continuing basis

***Simple Interest***

Interest = Principal • Rate • Time (or $I = PRT$)

Example 3: ***SIMPLE INTEREST*** How much do you pay if you borrow $100 for six months at 10%?

This does not account for the interest accrued on the interest, also known as Compound Interest.

***Compound Interest***

$$S = P\left(1+\frac{r}{n}\right)^{nt}$$

$S$ is the total money accrued, $P$ is the principal, $r$ is the annual percentage rate (APR), $n$ is the number of compounding periods per year, $t$ is time in years.

Example 4: ***SAVINGS*** How much money would you have to invest today at 6.5% APR—compounded monthly—to have $5,000 in ten years?

Example 5: ***SAVINGS*** How long will it take a $100 deposit to grow to $500 if it earns at 10% APR compounded daily?

## *TYPE III: FINANCES (SAVINGS AND LOAN ANNUITIES)*

An annuity is an account where you pay a specific amount periodically (monthly, weekly, etc.) instead of just one principal payment as you do with Compound Interest accounts. Annuities take two forms: savings and loans.

***Savings:*** $S = P \bullet \dfrac{\left(1+\dfrac{r}{n}\right)^{nt} - 1}{\left(\dfrac{r}{n}\right)}$    ***Loans:*** $L = P \bullet \dfrac{1-\left(1+\dfrac{r}{n}\right)^{-nt}}{\left(\dfrac{r}{n}\right)}$

$S$ is the amount saved, $L$ is the amount of the loan, $P$ is the periodic payment amount, $r$ is the annual percentage rate (APR), $n$ is the number of payments per year, $t$ is the time in years.

<u>Example 6:</u> ***LOAN*** You want to buy a $15,000 car. You can borrow the money at 4.9% APR for five years. What will the monthly payments be?

<u>Example 7:</u> ***SAVINGS ANNUITY.*** Suppose you want to buy a house in ten years and you can save $100 a month for the down payment. Suppose you find a savings fund that will earn 8% APR. How much money will you have for the down payment in ten years?

Section B2 Exercises

1. The number of bacteria in a culture is modeled by the function $n(t) = 500e^{0.45t}$, with $t$ in hours.
   a. What is the initial number of bacteria?
   b. How many bacteria are in the culture after three hours?
   c. How long will it take the number of bacteria in the culture to reach 10,000?

2. The mass $m(t)$ remaining after $t$ days from a 40g sample of thorium–234 is given by $m(t) = 40e^{-0.0277t}$.
   a. How much of the sample will remain after 60 days?
   b. After how long will only 10g of the sample remain?
   c. Physicists express the rate of decay of radioactive isotopes in terms of *half–life*, the time required for half the mass to decay. What is the half–life of thorium–234?

3. If the number of bacteria in a petri dish doubles every 30 minutes, how long will it take for the number to triple?

4. If a town's population doubles every ten years and is 20,000 this year, how long will it take the population to reach 100,000?

5. If a town's population decreases exponentially and drops from 50,000 to 44,000 between 1995 and 2005, what will the population be in 2020?

6. If 30% of a radioactive substance disappears in 15 years, what is the half–life of the substance?

7. How much would you have to invest today at 6.5% APR compounded monthly to have $10,000 in ten years?

8. How long will it take a $1,100 investment to grow to $10,000 if invested today at 10% APR compounded monthly?

9. Is it better to invest $300 for 10 years at 3.6% compounded monthly or at 4.1% compounded quarterly?

10. Suppose you want to buy a house in ten years and can save $200 a month for the down payment. You've found a mutual fund that will earn 4% APR. How much will you have in ten years?

11. Suppose you want to buy a house in ten years and will need $60,000 for the down payment. There is a mutual fund that will earn 1.3% APR. What will be the monthly investment?

12. Suppose you take a $240,000 house loan at 7% APR for 30 years. What are the monthly payments and how much will you actually pay the bank over the life of the loan?

13. Suppose you want to but a $40,000 car. You can borrow the money at 2.9% APR for five years. What will the monthly payments be?

14. Suppose you can afford a $250 monthly payments for a car and will borrow the money at 3.9% APR for five years. What is the maximum purchase price of the car?

15. Suppose you buy a $400,000 condo, making a down payment equal to 20% of the purchase price and taking out a loan for the rest. If the loan is at 4% fixed APR compounded monthly for 30 years, find the loan amount, the amount of each monthly payment, and the **total** amount paid for the condo over the course of the loan.

# Section B3—CONIC SECTIONS OVERVIEW

<u>*Vocabulary:*</u>
***Conic Section*** – a figure formed by the intersection of a plane and a double–napped cone (this means there are two cones "nose to nose"). The angle of the plane determines whether the conic section is a circle, ellipse, parabola, or hyperbola
***Traits*** – the characteristics of the graph of a function
***Domain*** – the set of all $x$–values that may be used in a function.
***Range*** – the set of all $y$–values that correspond to the $x$–values in the function's domain

The Conics share a common standard equation, namely,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Conics that have a $Bxy$ term are beyond the scope of this course. Assuming the conic is non-degenerate, identify its shape by looking at $A$ and $C$ in the <u>standard</u> equation as follows:

| | |
|---|---|
| Circle | $A = C$ |
| Parabola | $A = 0$ or $C = 0$ |
| Ellipse | $A$ and $C$ have the same sign, but $A \neq C$ |
| Hyperbola | $A$ and $C$ have different signs |

Additionally, each conic has a <u>general</u> (graphing) form that makes its traits more apparent.

| | |
|---|---|
| Circle | $(x-h)^2 + (y-k)^2 = r^2$ |
| Parabola | $(x-h)^2 = 4c(y-k)$ or $(y-k)^2 = 4c(x-h)$ |
| Ellipse | $\dfrac{(x-h)^2}{r_x^2} + \dfrac{(y-k)^2}{r_y^2} = 1$ |
| Hyperbola | $\dfrac{(x-h)^2}{r_x^2} - \dfrac{(y-k)^2}{r_y^2} = 1$ or $-\dfrac{(x-h)^2}{r_x^2} + \dfrac{(y-k)^2}{r_y^2} = 1$ |

Each conic will be explored separately; the meanings of $r_x$, $r_y$, $c$, $h$, $k$, and $r$ will be considered in future lessons. The main point is that, **while the shape can be determined from the Standard Form, the General (Graphing) Form is required to identify *traits* and to sketch.**

LEARNING OUTCOMES

- Identify a conic from an equation in standard form.
- Complete the square to change from standard to general form of a conic.
- Find the domain and range of a conic.

Example 1: Identify $x^2+y^2+4x-8y=0$ and find the general form

Solution: A = C, so this conic is a circle. Complete the square to convert an equation from standard to general form.

$$\begin{aligned} x^2+y^2+4x-8y &= 0 \\ x^2+4x+\_\_\_\_+y^2-8y+\_\_\_\_ &= 0+\_\_\_\_+\_\_\_\_ \\ x^2+4x+4+y^2-8y+16 &= 0+4+16 \\ (x+2)^2+(y-4)^2 &= 20 \end{aligned}$$

Example 2: Identify $x^2-4x-8y=0$ and find the general form

Solution: There is only one square, so this conic is a parabola

$$\begin{aligned} x^2-4x-8y &= 0 \\ x^2-4x+\_\_\_\_ &= 8y+\_\_\_\_ \\ x^2-4x+4 &= 8y+4 \\ (x-2)^2 &= 8\left(y+\frac{1}{2}\right) \end{aligned}$$

Example 3: Identify $x^2+3y^2+4x-12y-2=0$ and find the general form

Solution: $A \neq C$, but both are positive, so this conic is an ellipse. Factor out the 3 before completing the square in $y$.

$$\begin{aligned} x^2+3y^2+4x-12y-2 &= 0 \\ x^2+4x+3y^2-12y &= 2 \\ \left(x^2+4x+\_\_\_\_\right)+3\left(y^2-4y+\_\_\_\_\right) &= 2+\_\_\_\_+\_\_\_\_ \\ \left(x^2+4x+4\right)+3\left(y^2-4y+4\right) &= 2+4+12 \\ (x+2)^2+3(y-2)^2 &= 18 \\ \frac{(x+2)^2}{18}+\frac{(y-2)^2}{6} &= 1 \end{aligned}$$

Example 4: Identify $x^2-3y^2+4x-12y-2=0$ and find the general form

Solution: $A$ and $C$ have different signs ($C=-3$), so this is an hyperbola.

$$\begin{aligned}
x^2-3y^2+4x-12y-2 &= 0\\
x^2+4x-3y^2-12y &= 2\\
\left(x^2+4x+\underline{\quad}\right)-3\left(y^2+4y+\underline{\quad}\right) &= 2+\underline{\quad}-\underline{\quad}\\
\left(x^2+4x+4\right)-3\left(y^2+4y+4\right) &= 2+4-12\\
(x+2)^2-3(y+2)^2 &= -6\\
-\frac{(x+2)^2}{6}+\frac{(y+2)^2}{2} &= 1
\end{aligned}$$

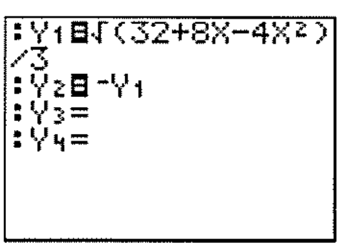
In this course, the conics are classified in terms of their geometric traits (center, radius, foci, etc.). Another way to look at conic sections is as a pair of irrational functions: this is the more common way to graph relations such as the conics in your calculator. These functions are found by isolating $y$ and using $\pm\sqrt{\ }$, instead of just the positive radical. To find domain and range, look at the graph of the conic. In order to do that, isolate $y$.

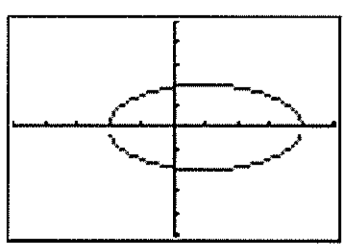
Example 5: Find the domain and range of $4x^2+9y^2-8x-32=0$

Solution:

$$\begin{aligned}
4x^2+9y^2-8x-32 &= 0\\
9y^2 &= 32+8x-4x^2\\
y^2 &= \frac{32+8x-4x^2}{9}\\
y &= \pm\frac{\sqrt{32+8x-4x^2}}{3}
\end{aligned}$$

Analysis of the graph shows that the domain is about –2 to 4 and the range is about –2 to 2. Use the *TRACE* key to verify these.



Mathematicians typically use interval notation, rather than inequality statements, to express domain and range, so domain is written as $x\in[-2,\ 4]$ and range is written as $y\in[-2,\ 2]$.

**Interval Notation:**

| | |
|---|---|
| Closed interval: | $x \in [a,\ b]$ means $a \le x \le b$ |
| Open interval: | $x \in (a,\ b)$ means $a < x < b$ |
| Half-open interval: | $x \in [a,\ b)$ means $a \le x < b$ |
| or | $x \in (a,\ b]$ means $a < x \le b$ |

Section B3 Exercises

For each of the following standard form equations:

(a) Identify each conic;
(b) convert to general (graphing) form;
(c) isolate $y$;
(d) graph on the calculator;
(e) state domain and range (use interval notation).

1. $x^2 + y^2 + 6x - 14y + 54 = 0$
2. $2x^2 + 2y^2 - 10x + 2y - 5 = 0$
3. $x^2 + 10x - 20y + 25 = 0$
4. $y^2 + 4x + 16y + 4 = 0$
5. $x^2 + 25y^2 + 6x - 100y + 9 = 0$
6. $9x^2 + 4y^2 + 36x - 8y + 4 = 0$
7. $x^2 - y^2 - 6y - 3 = 0$
8. $12x^2 - 4y^2 + 72x + 16y + 44 = 0$

# Section B4—CONICS II: THE PARABOLA

*Vocabulary:*
***Parabola*** – the set of all coplanar points equidistant from a fixed point and a fixed line
***Focus*** – the fixed point referred to in the parabola definition
***Directrix*** – the fixed line referred to in the parabola definition
***Latus Rectum*** – a segment parallel to the directrix and through the focus with endpoints on the parabola
***Eccentricity*** – a ratio: the distance from any point on the curve to the focus TO the distance from that point to the directrix

As stated in the overview, the general equation of a parabola is:

$$(x-h)^2 = 4c(y-k) \quad \text{or} \quad (y-k)^2 = 4c(x-h)$$

A parabola has the following traits:

1. Vertex = $(h,\ k)$

2. The direction it opens: The curve opens in the direction of the axis of the linear variable and the sign of $4c$. For example, $(y-2)^2 = -5(x-3)$ would open left because the $x$ is linear and the negative part of the $x$–axis is left.

3. Focus: $c$ units from the vertex, in the direction it opens

4. Directrix: The line $c$ units from the vertex, away from the direction it opens. It is $y = k - c$ or $x = h - c$, depending on which is the linear variable.

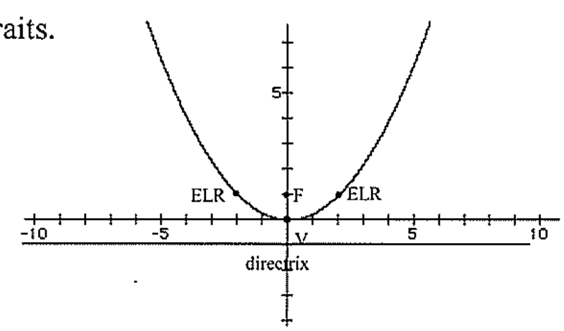
5. Endpoints of the Latus Rectum (ELRs): coordinate points $2c$ units from the focus on the parabola.

6. The Axis of Symmetry: The line through the vertex and focus.

7. Eccentricity: $e = 1$ for all parabolas.

**REMEMBER: The key to the traits is the variable that <u>doesn't have</u> the square.**

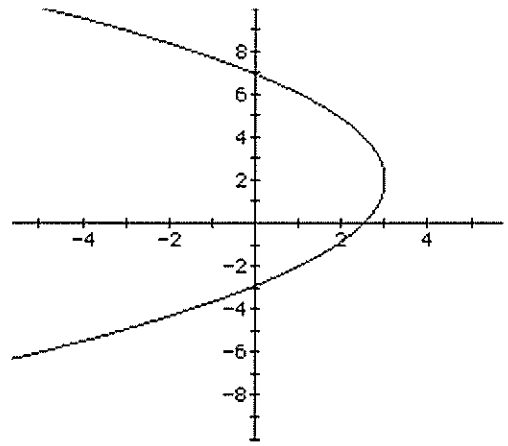
This sketch illustrates most of the parabola's traits.



| LEARNING OUTCOMES |
|---|
| Find all the traits and sketch a parabolic curve.<br>Find the equation of a parabola from its traits. |

<u>Example 1</u>: Find the traits and sketch $(y-2)^2 = -8(x-3)$

Solution:
(1) Vertex = (3 , 2)
(2) It opens left ($-x$).
(3) Focus: (1 , 2), because $c = 2$
(4) Directrix: $x = 5$ (note that the directrix is expressed in the form of a linear equation)
(5) ELRs: (1 , 6) and (1 , –2)
(6) Axis of Symmetry: $y = 2$
(7) $e = 1$.

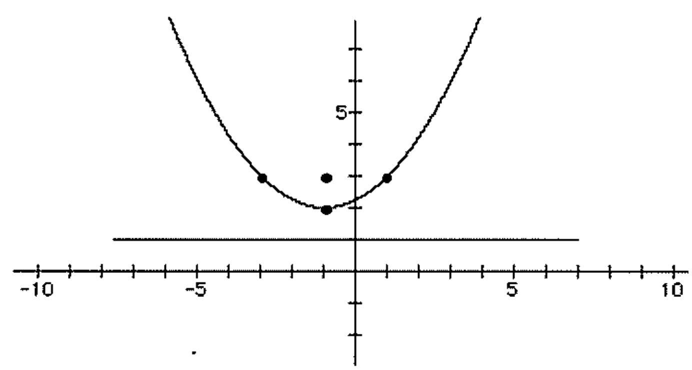



<u>Example 2</u>: Find the traits and sketch $x^2 + 2x - 4y + 9 = 0$

Solution: Convert the equation to general form first:

$$\begin{aligned} x^2+2x-4y+9 &= 0 \\ x^2+2x+\_\_\_\_ &= 4y-9+\_\_\_\_ \\ x^2+2x+1 &= 4y-9+1 \\ (x+1)^2 &= 4y-8 \\ (x+1)^2 &= 4(y-2) \end{aligned}$$

(1) Vertex = (–1 , 2)
(2) It opens up ($+y$).
(3) Focus: (–1 , 3), because $c = 1$
(4) Directrix: $y = -2$
(5) ELRs: (–3 , 3) and (1 , 3)
(6) Axis of Symmetry: $x = -1$
(7) $e = 1$.



Example 3: Find the equation of the parabola with vertex $(-3, 6)$ and directrix $y = 9$.

Solution:

The directrix is "$y =$ ", so the $y$ is the linear variable. The vertex is 3 units from the directrix, so $|c| = 3$ . The vertex is below the line, so the parabola opens down and therefore $c$ must be $-3$.

$$(x+3)^2 = -12(y-6)$$

Section B4 Exercises.

State the traits and sketch.

1. $(x-1)^2 = 8(y-2)$
2. $x^2 = -8(y-2)$
3. $(y+1)^2 = -4(x-3)$
4. $(x-4)^2 = 16y$
5. $x^2 + 4y + 4 = 0$
6. $y^2 - 12x - 4y - 44 = 0$
7. $x^2 + 2x + 12y + 37 = 0$
8. $y^2 - 4x + 8y - 28 = 0$

Find the equation of the parabola with these traits.

9. Vertex $(-1, 2)$ and Focus $(-1, 7)$
10. Vertex $(0, -2)$ and directrix $x = -3$
11. Focus $(0, 0)$ and directrix $y = 4$
12. Vertex $(2, 1)$, opens up, and LR length 8
13. Focus $(1, 5)$, ELR $(1, 0)$, and opens right
14. ELRs $(-1, 0)$ and $(-1, 30)$ and directrix $x = 14$

# Section B5—CONICS III: THE ELLIPSE

*Vocabulary:*

***Ellipse*** – (1) the set of all coplanar points the sum of whose distances to two fixed points is a constant (the "two focus property")
(2) the set of all coplanar points the ratio of whose distances to a fixed point and a fixed line is a constant $e$, where $0 < e < 1$ (the "focus – directrix property")
***Major Axis*** – the segment through the center and foci of the ellipse
***Minor Axis*** – the segment perpendicular to the major axis at the center
$r_x$ – distance from the center to the ellipse in the horizontal ($x$) direction; sometimes called the "$x$–radius"
$r_y$ – distance from the center to the ellipse in the vertical ($y$) direction; sometimes called the "$y$–radius"
***Focal Radius (c)*** – distance from the center to the focus
***Vertices*** – endpoints of the major axis
***Intercepts*** – endpoints of the minor axis
***Extent*** – domain of an ellipse

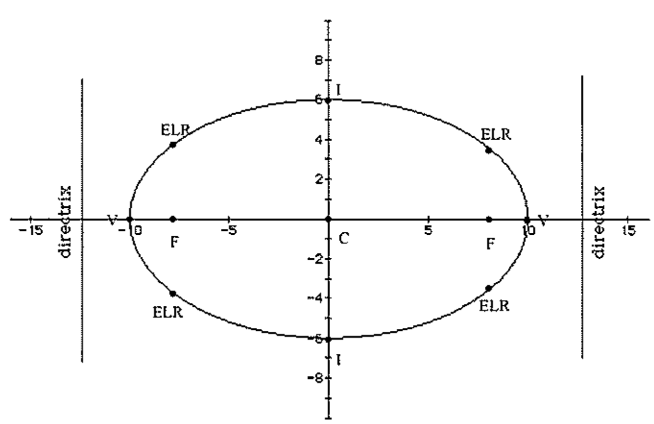
The general equation of an ellipse is:

$$\frac{(x-h)^2}{r_x^2}+\frac{(y-k)^2}{r_y^2}=1$$

Where $a = r_x$ or $r_y$, whichever is larger,
$b = r_x$ or $r_y$, whichever is smaller,
and $a^2 - b^2 = c^2$

An ellipse has the following traits:

1. Center: $(h, k)$
2. Vertices: $a$ from center on the major axis (indicated by the variable with the larger denominator).
3. Intercepts: $b$ from center on the minor axis
4. Foci: $c$ from center on the major axis
5. ELRs: $\frac{b^2}{a}$ from the foci
6. Directrices: $\frac{a^2}{c}$ from the center and parallel to the minor axis
7. Eccentricity: $\frac{c}{a}$; for an ellipse, $0 < e < 1$




**The key to the traits is the variable with the larger denominator.**

LEARNING OUTCOMES

Find all the traits and sketch an ellipse.

Find the equation of an ellipse from its traits.

Example 1: Find the traits and sketch $\frac{x^2}{9}+\frac{y^2}{25}=1$

Solution:

(1) Center: (0 , 0)

(2) Vertices: (0 , 5) and (0 , –5)

(3) Intercepts: (3 , 0) and (–3 , 0)

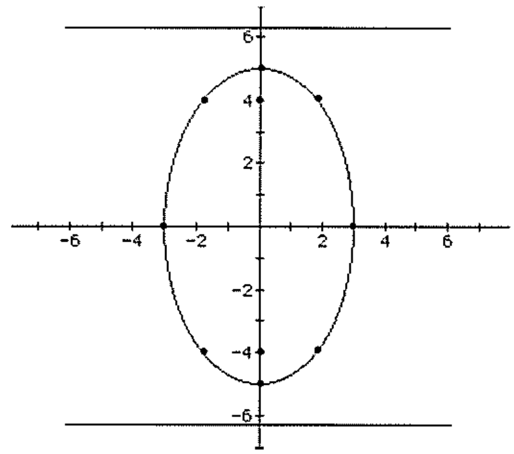
(4) Foci: (0 , 4) and (0 , –4).

Reasoning: $a^2-b^2=c^2 \to 25-9=16 \to c=\pm 4$

(5) ELRs: $\left(\frac{9}{5},4\right),\left(-\frac{9}{5},4\right),\left(\frac{9}{5},-4\right),\left(-\frac{9}{5},-4\right)$

(6) Directrices: $y=\pm\frac{25}{4}$

(7) Eccentricity: $e=\frac{4}{5}$




Example 2: List the traits and sketch $x^2+4y^2-4x+8y-8=0$

Solution:

$$
\begin{aligned}
x^2+4y^2-4x+8y-8 &= 0\\
x^2-4x+4y^2+8y &= 8\\
\left(x^2-4x+\_\_\_\_\right)+4\left(y^2+2y+\_\_\_\_\right) &= 8+\_\_\_\_+\_\_\_\_\\
\left(x^2-4x+4\right)+4\left(y^2+2y+1\right) &= 8+4+4\\
(x-2)^2+4(y+1)^2 &= 16\\
\frac{(x-2)^2}{16}+\frac{(y+1)^2}{4} &= 1
\end{aligned}
$$

(1) Center: (2 , –1)

(2) Vertices: (6 , –1) and (–2 , –1)

(3) Intercepts: (2 , 1) and (2 , –3)

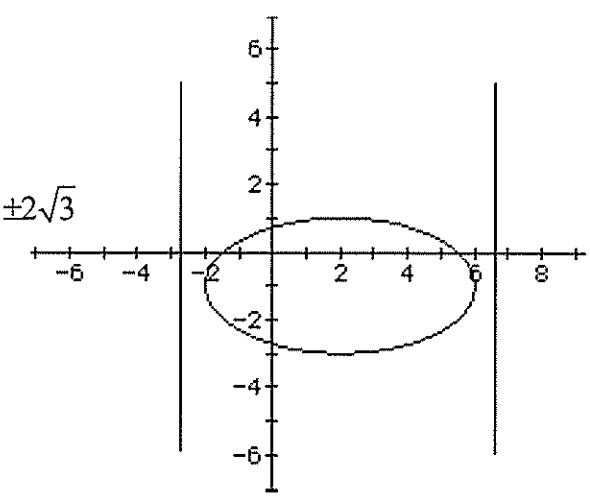
(4) Foci: $\left(2\pm 2\sqrt{3},-1\right)$

Reasoning: $a^2-b^2=c^2 \to 16-4=12 \to c=\pm 2\sqrt{3}$

(5) ELRs: $\left(2\pm 2\sqrt{3},0\right)$ and $\left(2\pm 2\sqrt{3},-2\right)$

(6) Directrices: $x=2\pm\frac{16}{2\sqrt{3}}$ or $x=2\pm\frac{8}{\sqrt{3}}$

(7) Eccentricity: $e=\frac{\sqrt{3}}{2}$



Example 3: Find the equation of the ellipse with intercepts (4, –1) and (0, –1) and vertices (2, –5) and (2, 3).

Solution:

The center must be halfway between the vertices, so the center is (2 , –1). We can see that the distance from the center to the intercepts is 2, so $b = 2$ and the distance from the center to the vertices is 4, so $a = 4$. The vertices are aligned vertically from one another, so the major axis is the $y$–axis. Therefore,

$$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{16}=1$$

Example 4: Find the equation of the ellipse with Foci (20, 0) and (0, 0) and $e=\frac{5}{13}$.

Solution:

Because of the foci, the center is (10, 0), and the $x$ axis is the major axis. $e=\frac{5}{13}=\frac{c}{a}$ , but the foci show that $c = 10$. So $a = 26$, and by $a^2-b^2=c^2$, $b=24$. Therefore,

$$\frac{(x-10)^2}{26^2}+\frac{y^2}{24^2}=1$$

or, $$\frac{(x-10)^2}{676}+\frac{y^2}{576}=1$$

Section B5 Exercises

List the traits and sketch.

1. $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

2. $\dfrac{(x+2)^2}{9}+\dfrac{(y+1)^2}{25}=1$

3. $\dfrac{(x-1)^2}{289}+\dfrac{(y-1)^2}{64}=1$

4. $\dfrac{x^2}{169}+\dfrac{y^2}{144}=1$

5. $9x^2+4y^2-72x-24y+144=0$

6. $x^2+4y^2-16=0$

7. $3x^2+2y^2-12=0$

8. $16x^2+25y^2+160x+200y+400=0$

Find the equations of the ellipse with these traits.

9. Vertices (–1, 2) and (–1, 12) and Focus (–1, 10)

10. Vertices (0, –2) and (6, –2) and Intercept (3, 0)

11. Center (0, 0) and directrix $x=\dfrac{25}{4}$

12. Foci (2, 0) and (8, 0) and LR length $\dfrac{32}{5}$

13. Foci (1, 5) and (1, –7) and $e=\dfrac{3}{5}$

# Section B6—GRAPHING POLYNOMIAL FUNCTIONS USING THE CALCULATOR

*Vocabulary*:
***Degree*** – the maximum number of factors that appear as variables in any one term. May be expressed as a number (3, 4) or a word (cubic, quartic).
***Zeros**** *(x-intercepts)* – the points where the curve crosses the $x$-axis
***Domain*** – the set of all $x$–values that may be used in a function.
***Range*** – the set of all $y$–values that correspond to the $x$–values in the function's domain
***Extreme Points*** – high and low points on the curve
***Maximum Point*** – the high point (plural: *maxima*)
***Minimum Point*** – the low point (plural: *minima*)
***Relative vs. Absolute Extreme Points*** – relative refers to points at all the crests and troughs of the curve; absolute refers to the highest or lowest points for the entire function

*In most math texts, "zero" or "$x$-intercept" only means the $x$–coordinate, but the calculator shows both coordinates. To be consistent in the context of graphing, use the above definition.

LEARNING OUTCOME:
Use a graphing calculator to find the traits and sketch an accurate graph of a polynomial function

Example 1: List all traits and sketch $f(x) = x^2 + 2x - 5$

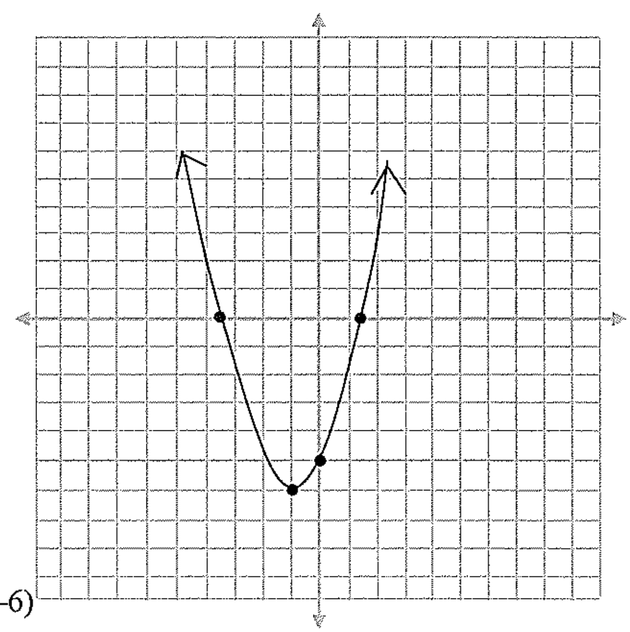
Solution:

degree: the $x^2$ term defines this function as degree 2 (quadratic)

zeros: $x^2 + 2x - 5 = 0$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-5)}}{2(1)}$$

$x = 1.449$ or $-3.449$
Zeros are (1.449, 0) and (–3.449, 0)



y–intercept: (0, –5)

minima (type): absolute minimum at (–1, –6)

maxima (type): none

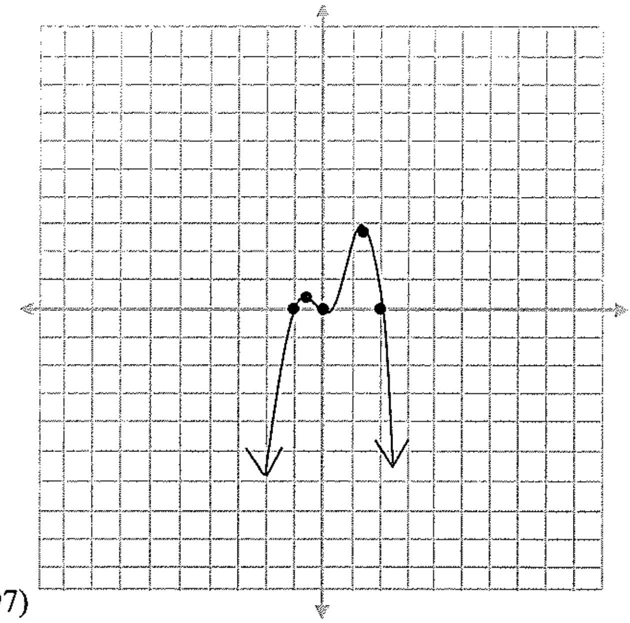
domain: $x \in \mathbb{R}$ or $x \in (-\infty, \infty)$

range: $y \in [-6, \infty)$

**NOTE:**
Graphing window $[-10, 10]\text{x}[-10, 10]$ was used; note that all points are **visible**.

Example 2: List all traits and sketch $f(x) = -x^4 + x^3 + 2x^2$

Solution: use a graphing window of $[-10,\ 10] \text{x} [-10,\ 10]$ . Be sure all *points* are VISIBLE on the sketch.



degree: the $x^4$ term defines this function as degree 4 (quartic)

zeros:

$-x^4 + x^3 + 2x^2 = 0$

$-x^2\left(x^2 - x - 2\right) = 0$

$-x^2(x-2)(x+1) = 0$

$x = 0,\ 2,\ \text{or}\ -1$

Zeros are (0, 0), (2, 0) and (–1, 0)

y–intercept: (0, 0)

minima (type): relative minimum at (0, 0)

maxima (type): relative maximum at (–0.693, 0.397)
absolute maximum at (1.443, 2.833)

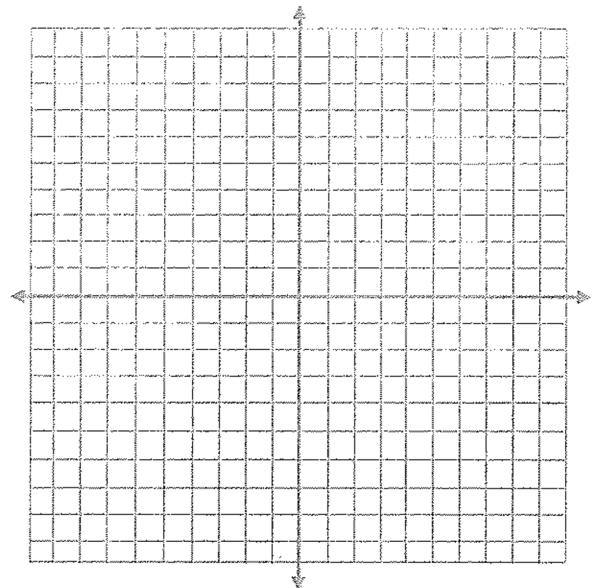
domain: $x \in \mathbb{R}$ or $x \in (-\infty,\ \infty)$

range: $y \in (-\infty,\ 2.833]$

## Section B6 Exercises

For each polynomial function, fill in the blanks to describe the degree, state the $x$–intercepts and the $y$–intercepts to the nearest thousandth, the minima and maxima (specify either relative or absolute for each) to the nearest thousandth, and the domain and range using interval notation. Neatly sketch each function using the coordinate planes that have been provided. Scale axes as needed; points must be visible on each sketch!!

1. $f(x) = x^4 - 4x^3 + 2x^2 + x + 4$



degree:

$x$–intercept(s):

y–intercept:

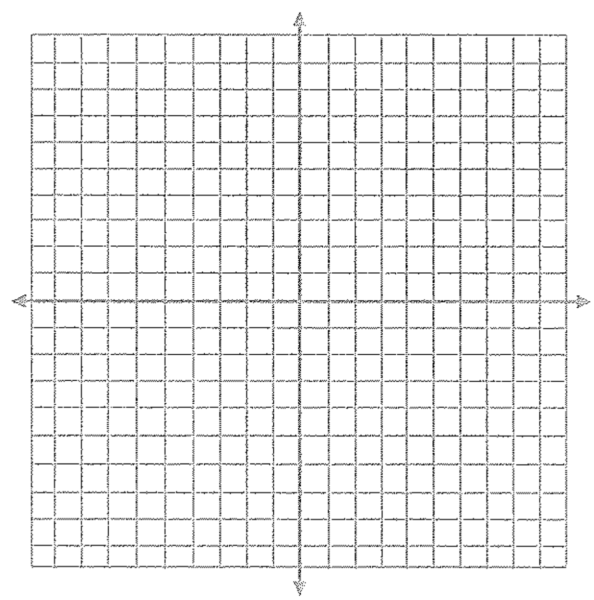
minima (type):

maxima (type):

domain:

range:

2. $f(x) = x^3 + x^2 - x - 2$



degree:

$x$–intercept(s):

y–intercept:

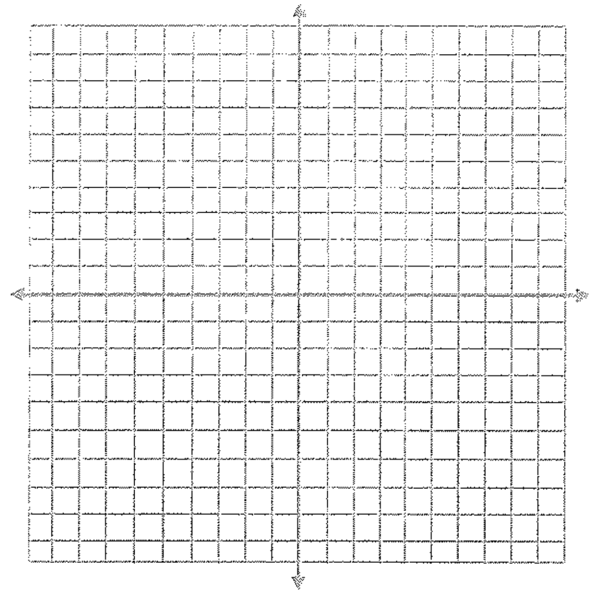
minima (type):

maxima (type):

domain:

range:

3. $f(x) = x^5 - 4x^3 + 4x - 1$



degree:

$x$–intercept(s):

y–intercept:

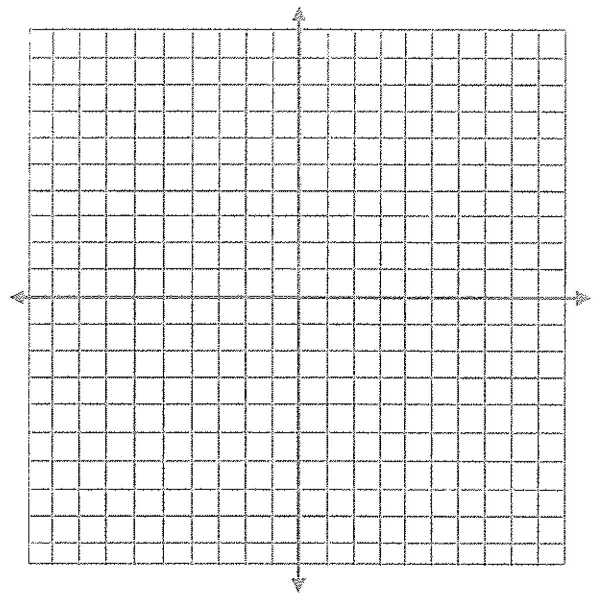
minima (type):

maxima (type):

domain:

range:

4. $f(x) = x^3 + 11x^2 + 35x + 32$



degree:

$x$–intercept(s):

y–intercept:

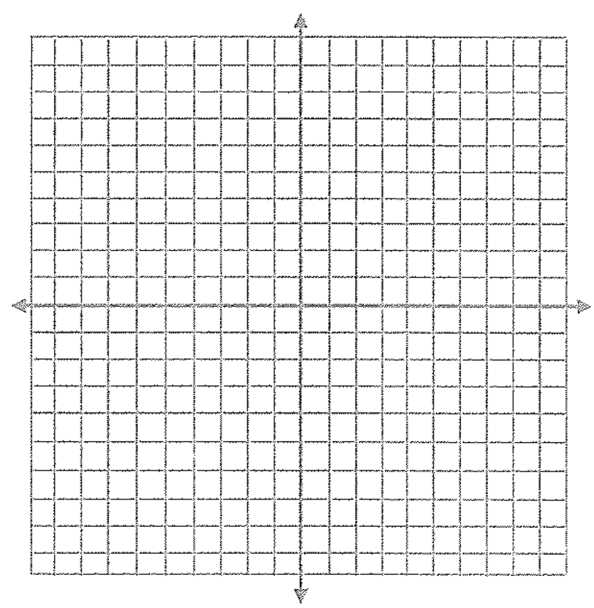
minima (type):

maxima (type):

domain:

range:

5. $f(x) = -x^5 + 4x^3 - 5x - 2$



degree:

$x$–intercept(s):

y–intercept:

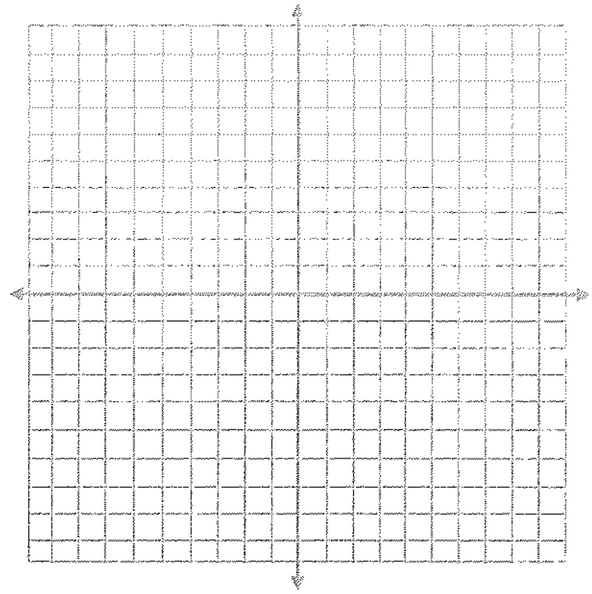
minima (type):

maxima (type):

domain:

range:

6. $f(x) = x^4 - x^2 + x$



degree:

$x$–intercept(s):

y–intercept:

minima (type):

maxima (type):

domain:

range:

# Appendix

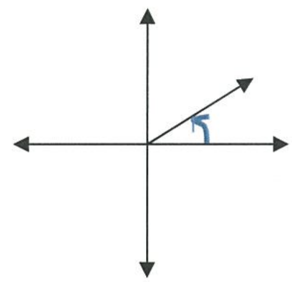
# ANSWERS TO EXERCISES

## Section A1

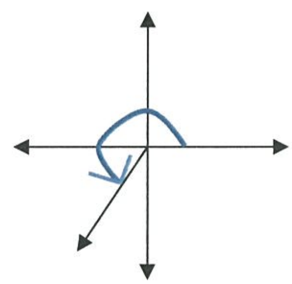
1. $\frac{\pi}{6}$
2. $\frac{\pi}{12}$
3. $\frac{5\pi}{9}$
4. $\frac{10\pi}{9}$
5. $\frac{5\pi}{12}$
6. $\frac{7\pi}{12}$
7. $\frac{2\pi}{3}$
8. $\frac{4\pi}{3}$
9. $-\frac{16\pi}{9}$
10. $-\frac{25\pi}{18}$
11. $-\frac{17\pi}{36}$
12. $-\frac{35\pi}{36}$
13. $\left(\frac{180}{\pi}\right)^{\circ}$
14. $\left(\frac{360}{\pi}\right)^{\circ}$
15. 1440°
16. –2160°
17. 135°
18. –495°
19. $\frac{20\pi}{3}$ cm
20. 10.5 feet

## Section A2

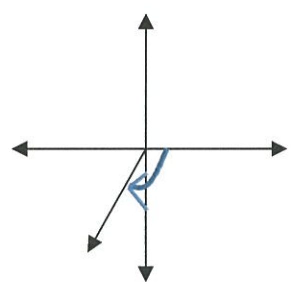
1.



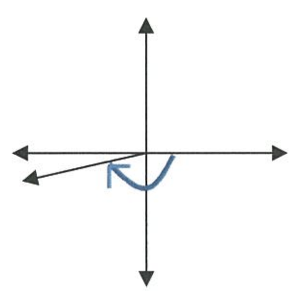
2.



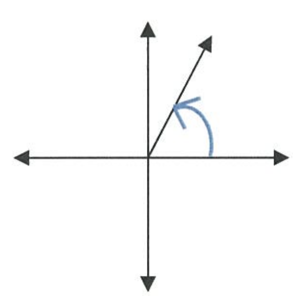
3.



4.

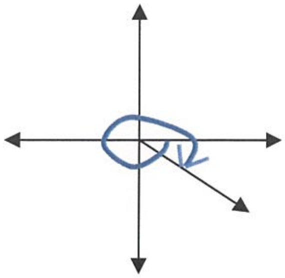


5.

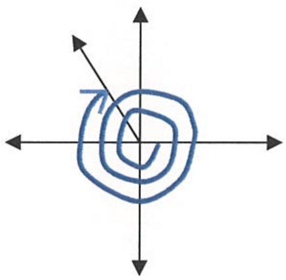


Section A2 (continued)

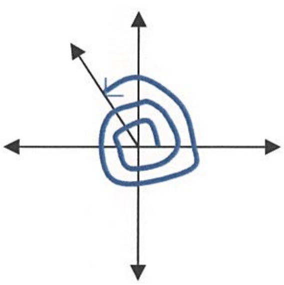
6.



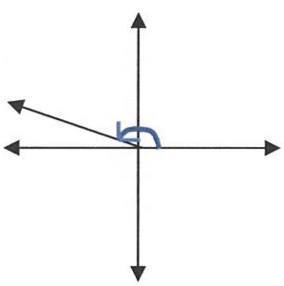
7.



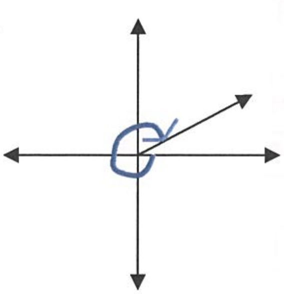
8.



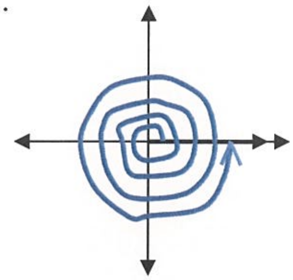
9.



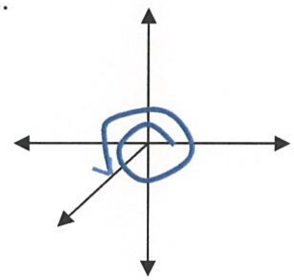
10.



11.



12.



13. 45°
14. 80°
15. 60°
16. $\frac{\pi}{6}$
17. $\frac{\pi}{3}$
18. $\frac{\pi}{4}$

Section A3

1. See Unit Circle
2. See Table
3. $\frac{1}{2}$
4. 1
5. $-\frac{8}{3}$
6. 1
7. 1
8. 1
9. 0
10. $2\sqrt{2}$

Section A4

1. $\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$

$\tan\alpha = \frac{4}{3}$

$\csc\alpha = \frac{5}{4}$

$\sec\alpha = \frac{5}{3}$

$\cot\alpha = \frac{3}{4}$

2. $\sin\alpha = -\frac{4}{\sqrt{17}}$

$\cos\alpha = -\frac{1}{\sqrt{17}}$

$\tan\alpha = 4$

$\csc\alpha = -\frac{\sqrt{17}}{4}$

$\sec\alpha = -\sqrt{17}$

$\cot\alpha = \frac{1}{4}$

3. $\sin\alpha = -\frac{3}{\sqrt{13}}$

$\cos\alpha = \frac{2}{\sqrt{13}}$

$\tan\alpha = -\frac{3}{2}$

$\csc\alpha = -\frac{\sqrt{13}}{3}$

$\sec\alpha = \frac{\sqrt{13}}{2}$

$\cot\alpha = -\frac{2}{3}$

4. $\sin\alpha = \frac{1}{\sqrt{2}}$

$\cos\alpha = -\frac{1}{\sqrt{2}}$

$\tan\alpha = -1$

$\csc\alpha = \sqrt{2}$

$\sec\alpha = -\sqrt{2}$

$\cot\alpha = -1$

5. $\cos A = \frac{\sqrt{5}}{3}$

$\tan A = \frac{2}{\sqrt{5}}$

$\csc A = \frac{3}{2}$

$\sec A = \frac{3}{\sqrt{5}}$

$\cot A = \frac{\sqrt{5}}{2}$

6. $\sin A = -\frac{\sqrt{65}}{9}$

$\tan A = \frac{\sqrt{65}}{4}$

$\csc A = -\frac{9}{\sqrt{65}}$

$\sec A = -\frac{9}{4}$

$\cot A = \frac{4}{\sqrt{65}}$

7. $\sin B = -\frac{25}{\sqrt{1201}}$

$\cos B = -\frac{24}{\sqrt{1201}}$

$\csc B = -\frac{\sqrt{1201}}{25}$

$\sec B = -\frac{\sqrt{1201}}{24}$

$\cot B = \frac{24}{25}$

Section A4 (continued)

8. $\sin\phi = -\frac{5}{6}$

$\cos\phi = \frac{\sqrt{11}}{6}$

$\tan\phi = -\frac{5}{\sqrt{11}}$

$\sec\phi = \frac{6}{\sqrt{11}}$

$\cot\phi = -\frac{\sqrt{11}}{5}$

9. $\sin\varphi = \frac{2\sqrt{7}}{\sqrt{37}}$

$\cos\varphi = -\frac{3}{\sqrt{37}}$

$\tan\varphi = -\frac{2\sqrt{7}}{3}$

$\csc\varphi = \frac{\sqrt{37}}{2\sqrt{7}}$

$\cot\varphi = -\frac{3}{2\sqrt{7}}$

10. $\sin\omega = \frac{60}{61}$

$\cos\omega = -\frac{11}{61}$

$\tan\omega = -\frac{60}{11}$

$\csc\omega = \frac{61}{60}$

$\sec\omega = -\frac{61}{11}$

11. 0.242
12. 0.284
13. –0.839
14. 0.138
15. 1.041
16. 3.864
17. $\begin{Bmatrix} 40.693^\circ \pm 360^\circ n \\ 139.307^\circ \pm 360^\circ n \end{Bmatrix}$
18. $\begin{Bmatrix} 121.399^\circ \pm 360^\circ n \\ -121.399^\circ \pm 360^\circ n \end{Bmatrix}$
19. $\{55.072^\circ \pm 180^\circ n\}$
20. $\{-56.896^\circ \pm 180^\circ n\}$
21. $\begin{Bmatrix} 55.842^\circ \pm 360^\circ n \\ -55.842^\circ \pm 360^\circ n \end{Bmatrix}$
22. DNE
23. DNE
24. $\begin{Bmatrix} 1.672 \pm 2\pi n \\ -1.672 \pm 2\pi n \end{Bmatrix}$
25. $\{1.347 \pm \pi n\}$
26. $\{-1.368 \pm \pi n\}$
27. $\begin{Bmatrix} 1.182 \pm 2\pi n \\ -1.182 \pm 2\pi n \end{Bmatrix}$
28. $\begin{Bmatrix} 1.024 \pm 2\pi n \\ 2.118 \pm 2\pi n \end{Bmatrix}$

Section A5

Note: Complete solutions are not provided here, as multiple approaches yield the desired result. A hint has been provided to assist in beginning each proof.

1. Convert to sine and cosine
2. Separate the fraction; convert to sine and cosine
3. Factor; Pythagorean identity
4. Multiply; Pythagorean identities
5. Multiply
6. Multiply
7. Separate the fraction
8. Separate the fraction
9. Multiply
10. Distribute; convert to sine and cosine
11. Common denominator
12. Common denominator
13. Separate the fraction
14. Convert to sine and cosine
15. Common denominator
16. Factor sum of cubes
17. Factor sum of cubes
18. Pythagorean identity; factor
19. Common denominator
20. Factor difference of cubes; Pythagorean identity

Section A6

1. $-12.818\vec{i}+54.375\vec{j}$
   55.865 units at 103.264°
2. $-6.134\vec{i}-2.024\vec{j}$
   6.459 units at –161.738°
3. $791.330\vec{i}+320.570\vec{j}$
   853.796 units at 22.053°
4. $8.530\vec{i}+0.617\vec{j}$
   8.552 units at 4.137°
5. 403.550 mph at –152.045°
6. 26.977 knots at 173.982°

Section A7

1. 58.761
2. 53.299
3. 26.384°, 36.336°, 117.280°
4. 16.598°
5. $\vec{r}=19.528;\ \theta=80.169°$
6. $\vec{r}=26.116;\ \theta=103.051°$
7. $\vec{r}=14.664;\ \theta=45.838°$
8. $\vec{r}=46.751;\ \theta=7.153°$

Section A8

1. $a=5.098,\ c=4.065,\ A=10.235$ units$^2$
2. $d=4.634,\ e=5.908,\ A=13.556$ units$^2$
3. $k=5.374,\ l=5.275,\ A=11.750$ units$^2$
4. $d=17.433,\ p=54.471,\ A=358.328$ units$^2$
5. $b=8.751, f=3.411,\ A=7.236$ units$^2$
6. $c=13.116,\ r=7.143,\ A=39.289$ units$^2$
7. $A=10\sqrt{2}$ units$^2$
8. DNE
9. $A=44.466$ units$^2$

Section A9

1. $m\angle R=74.949°,\ m\angle L=74.051°,\ l=14.935$
   OR
   $m\angle R=105.051°,\ m\angle L=43.949°,\ l=10.78$
2. $m\angle F=24.655°,\ m\angle E=120.345°,\ e=16.550$
3. $m\angle M=10.529°,\ m\angle L=6.471°,\ l=3.084$
4. No such triangle
5. $m\angle C=43.495°,\ m\angle F=101.505°, f=8.542$
   OR
   $m\angle C=136.505°,\ m\angle F=8.495°, f=1.288$
6. $m\angle C=60°,\ m\angle Q=90°,\ c=5\sqrt{3}$
7. $m\angle I=35.538°,\ m\angle A=116.065°,\ m\angle J=28.39$
8. $m\angle R=73.283°,\ m\angle K=30.717°,\ l=15.197$

Section A10

1. 120.796 feet
2. 151.411 yards and 162.717 yards
3. 41.562; $A=294.522$ units$^2$
4. 1.699 miles
5. $\sin^{-1}(1.272)=$ DNE; 40.688 feet, 12.288 feet
6. 7515.815 feet$^2$
7. 1.229 miles and 8.947 miles
8. 1605.036 feet, 92.713°, 82.287°, 1965361.327 square feet

Section B1

1. $\ln 12$
2. $\log 4$
3. $\log_5\left(\frac{1}{25}\right)$, which is –2.
4. $2\log x-\frac{1}{3}\log w-\frac{1}{2}\log y+3\log z$
5. $3\log x-\frac{2}{3}\log w+\frac{2}{5}\log y-4\log z$
6. $-\frac{1}{3}\log x+4\log y+3\log w-\frac{2}{5}\log z$
7. –4
8. –6
9. 5
10. 4
11. $\frac{1}{25}$
12. 1726.999 or 1.001
13. 0.477

Section B1 (continued)

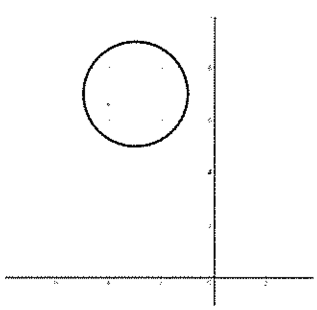
14. 1 or $\frac{1}{2}$
15. $\frac{2}{3}$
16. 2.094 or $-1.267$

Section B2

1. (a) 500
   (b) 1928
   (c) 6.657 hours
2. (a) 7.590 grams
   (b) 50.047 days
   (c) 25.023 days
3. 47.549 minutes
4. 23.219 years
5. 36322 people
6. 29.150 years
7. \$5,229.62
8. 22.165 years
9. 4.1% compounded quarterly
10. \$29,949.96
11. \$468.47
12. \$1,596.73; \$574,821.36
13. \$716.97
14. \$13,608.09
15. \$320,000.; \$\$1,527.73; \$629,982.42

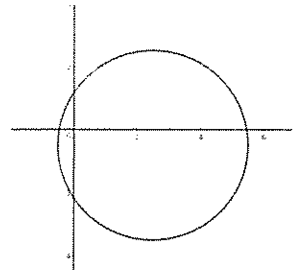
Section B3

1. (a) Circle
   (b) $(x+3)^2+(y-7)^2=4$
   (c) $y=7\pm\sqrt{4-(x+3)^2}$
   (d)



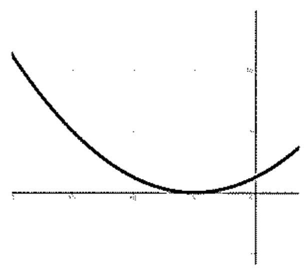
   (e) $x\in[-5,-1];\ y\in[5,9]$

2. (a) Circle
   (b) $\left(x-\frac{5}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=9$
   (c) $y=-\frac{1}{2}\pm\sqrt{9-\left(x-\frac{5}{2}\right)^2}$
   (d)



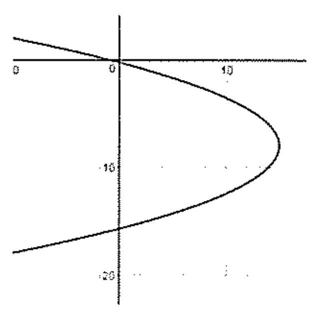
   (e) $x\in\left[-\frac{1}{2},\frac{11}{2}\right];\ y\in\left[-\frac{7}{2},\frac{5}{2}\right]$

3. (a) Parabola
   (b) $(x+5)^2=20y$
   (c) $y=\frac{(x+5)^2}{20}$
   (d)



   (e) $x\in(-\infty,\infty);\ y\in[0,\infty)$

4. (a) Parabola
   (b) $(y+8)^2=-4(x-15)$
   (c) $y=-8\pm\sqrt{-4(x-15)}$
   (d)




   (e) $x\in(-\infty,15];\ y\in(-\infty,\infty)$

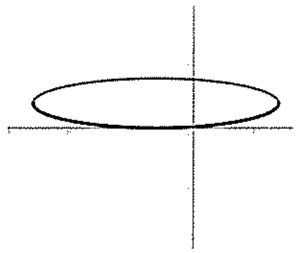
Section B3 (continued)

5. (a) Ellipse

(b) $\frac{(x+3)^2}{100}+\frac{(y-2)^2}{4}=1$

(c) $y=2\pm\sqrt{\frac{100-(x+3)^2}{25}}$

(d)



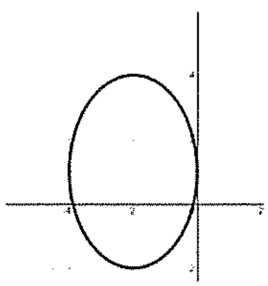
(e) $x\in[-13,7]$; $y\in[0,4]$

6. (a) Ellipse

(b) $\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}=1$

(c) $y=1\pm\sqrt{\frac{36-9(x+2)^2}{4}}$

(d)



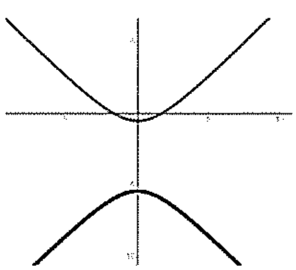
(e) $x\in[-4,0]$; $y\in[-2,4]$

7. (a) Hyperbola

(b) $-\frac{x^2}{6}+\frac{(y+3)^2}{6}=1$

(c) $y=-3\pm\sqrt{6+x^2}$

(d)



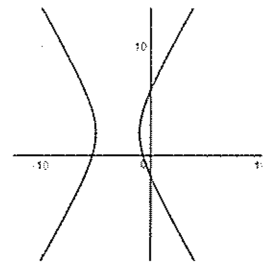
(e) $x\in(-\infty,\infty)$;
$y\in(-\infty,-5.449]\cup[-0.551,\infty)$

8. (a) Hyperbola

(b) $\frac{(x+3)^2}{4}-\frac{(y-2)^2}{12}=1$

(c) $y=2\pm\sqrt{-12+3(x+3)^2}$

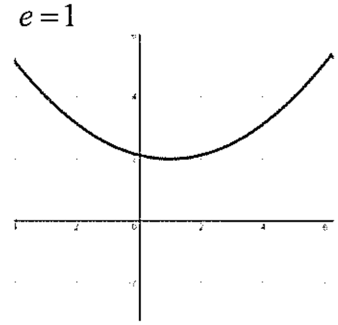
(d)



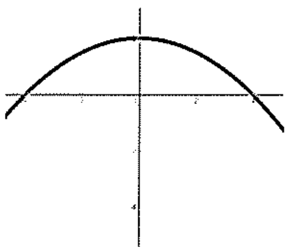
(e) $x\in(-\infty,-5]\cup[-1,\infty)$; $y\in(-\infty,\infty)$

Section B4

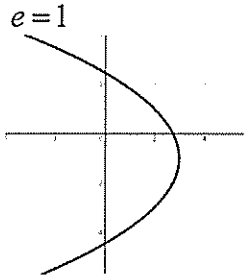
1. (1) Vertex: $(1,2)$
   (2) opens up
   (3) Focus: $(1,4)$
   (4) directrix: $y=0$
   (5) ELRs: $(-3,4)$, $(5,4)$
   (6) Axis of Symmetry: $x=1$
   (7) $e=1$



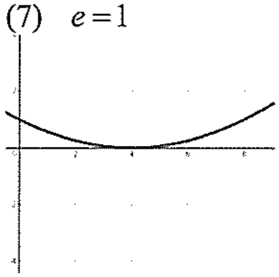
2. (1) Vertex: $(0,2)$
   (2) opens down
   (3) Focus: $(0,0)$
   (4) directrix: $y=4$
   (5) ELRs: $(4,0)$, $(-4,0)$
   (6) Axis of Symmetry: $x=0$
   (7) $e=1$



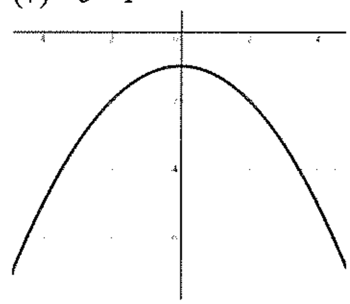
3. (1) Vertex: $(3,-1)$
   (2) opens left
   (3) Focus: $(2,-1)$
   (4) directrix: $x=0$
   (5) ELRs: $(2,1)$, $(2,-3)$
   (6) Axis of Symmetry: $y=-1$
   (7) $e=1$



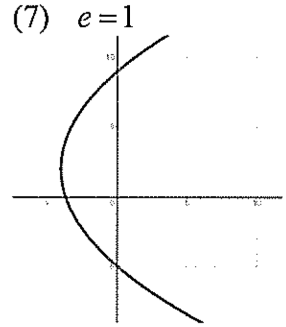
4. (1) Vertex: $(4,0)$
   (2) opens up
   (3) Focus: $(4,4)$
   (4) directrix: $y=-4$
   (5) ELRs: $(12,4)$, $(-4,4)$
   (6) Axis of Symmetry: $x=4$
   (7) $e=1$



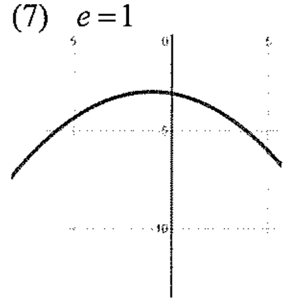
5. (1) Vertex: $(0,-1)$
   (2) opens down
   (3) Focus: $(0,-2)$
   (4) directrix: $y=0$
   (5) ELRs: $(2,-2)$, $(-2,-2)$
   (6) Axis of Symmetry: $x=0$
   (7) $e=1$



6. (1) Vertex: $(-4,2)$
   (2) opens right
   (3) Focus: $(-1,2)$
   (4) directrix: $x=-7$
   (5) ELRs: $(-1,8)$, $(-1,-4)$
   (6) Axis of Symmetry: $y=2$
   (7) $e=1$

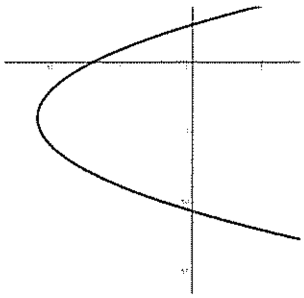


7. (1) Vertex: $(-1,-3)$
   (2) opens down
   (3) Focus: $(-1,-6)$
   (4) directrix: $y=0$
   (5) ELRs: $(-7,-6)$, $(5,-6)$
   (6) Axis of Symmetry: $x=-1$
   (7) $e=1$



Section B4 (continued)

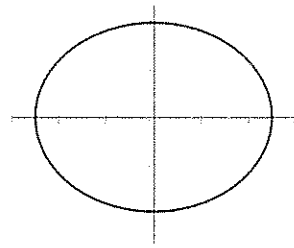
8. (1) Vertex: $(-11,-4)$
   (2) opens right
   (3) Focus: $(-10,-4)$
   (4) directrix: $x=-12$
   (5) ELRs: $(-10,-2)$, $(-10,-6)$
   (6) Axis of Symmetry: $y=-4$
   (7) $e=1$



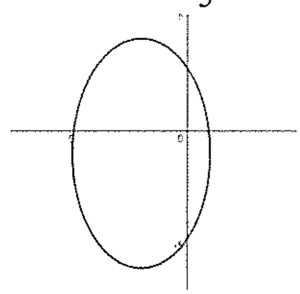
9. $(x+1)^2=20(y-2)$
10. $(y+2)^2=-12x$
11. $x^2=-8y$
12. $(x-2)^2=8(y-1)$
13. $(y-5)^2=10\left(x+\frac{3}{2}\right)$
14. $(y-15)^2=-30\left(x-\frac{13}{2}\right)$

Section B5

1. Center: $(0,0)$
   Vertices: $(5,0)$, $(-5,0)$
   Intercepts: $(0,4)$, $(0,-4)$
   Foci: $(3,0)$, $(-3,0)$
   ELRs:
   $\left(3,\frac{16}{5}\right)$, $\left(3,-\frac{16}{5}\right)$, $\left(-3,\frac{16}{5}\right)$, $\left(-3,-\frac{16}{5}\right)$
   Directrices: $x=\frac{25}{3}$, $x=-\frac{25}{3}$
   Eccentricity: $\frac{3}{5}$



2. Center: $(-2,1)$
   Vertices: $(-2,4)$, $(-2,-6)$
   Intercepts: $(-5,-1)$, $(1,-1)$
   Foci: $(-2,3)$, $(-2,-5)$
   ELRs:
   $\left(-\frac{19}{5},3\right)$, $\left(-\frac{1}{5},3\right)$, $\left(-\frac{19}{5},-5\right)$, $\left(-\frac{1}{5},-5\right)$
   Directrices: $y=\frac{21}{4}$, $y=-\frac{29}{4}$
   Eccentricity: $\frac{4}{5}$



3. Center: $(1,1)$
   Vertices: $(18,1)$, $(-16,1)$
   Intercepts: $(1,9)$, $(1,-7)$
   Foci: $(16,1)$, $(-14,1)$
   ELRs:
   $\left(16,\frac{81}{17}\right)$, $\left(16,-\frac{47}{17}\right)$, $\left(-14,\frac{81}{17}\right)$, $\left(-14,-\frac{47}{17}\right)$
   Directrices: $x=\frac{304}{15}$, $x=-\frac{274}{15}$
   Eccentricity: $\frac{15}{17}$

<u>Section B5</u> (continued)

4. Center: $(0,0)$

   Vertices: $(13,0)$, $(-13,0)$

   Intercepts: $(0,12)$, $(0,-12)$

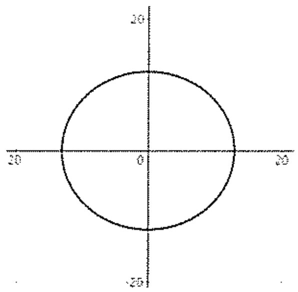
   Foci: $(5,0)$, $(-5,0)$

   ELRs:

   $\left(5,\frac{144}{13}\right)$, $\left(5,-\frac{144}{13}\right)$, $\left(-5,\frac{144}{13}\right)$, $\left(-5,-\frac{144}{13}\right)$

   Directrices: $x=\frac{169}{5}$, $x=-\frac{169}{5}$

   Eccentricity: $\frac{5}{13}$



5. Center: $(4,3)$

   Vertices: $(4,6)$, $(4,0)$

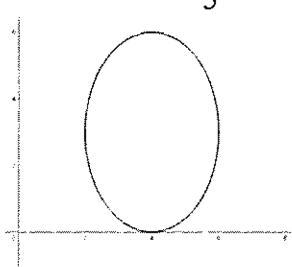
   Intercepts: $(2,3)$, $(6,3)$

   Foci: $\left(4,3+\sqrt{5}\right)$, $\left(4,3-\sqrt{5}\right)$

   ELRs: $\left(\frac{16}{3},3+\sqrt{5}\right)$, $\left(\frac{8}{3},3+\sqrt{5}\right)$, $\left(\frac{16}{3},3-\sqrt{5}\right)$, $\left(\frac{8}{3},3-\sqrt{5}\right)$

   Directrices: $y=3+\frac{9}{\sqrt{5}}$, $y=3-\frac{9}{\sqrt{5}}$

   Eccentricity: $\frac{\sqrt{5}}{3}$



6. Center: $(0,0)$

   Vertices: $(4,0)$, $(-4,0)$

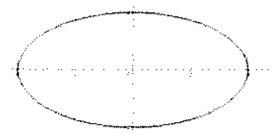
   Intercepts: $(0,2)$, $(0,-2)$

   Foci: $\left(2\sqrt{3},0\right)$, $\left(-2\sqrt{3},0\right)$

   ELRs: $\left(2\sqrt{3},1\right)$, $\left(2\sqrt{3},-1\right)$, $\left(-2\sqrt{3},1\right)$, $\left(2\sqrt{3},-1\right)$

   Directrices: $x=\frac{8}{\sqrt{3}}$, $x=-\frac{8}{\sqrt{3}}$

   Eccentricity: $\frac{\sqrt{3}}{2}$



7. Center: $(0,0)$

   Vertices: $\left(0,\sqrt{6}\right)$, $\left(0,-\sqrt{6}\right)$

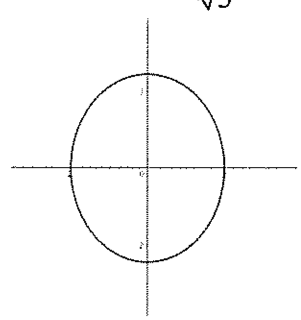
   Intercepts: $(2,0)$, $(-2,0)$

   Foci: $\left(0,\sqrt{2}\right)$, $\left(0,-\sqrt{2}\right)$

   ELRs: $\left(\frac{4}{\sqrt{6}},\sqrt{2}\right)$, $\left(-\frac{4}{\sqrt{6}},\sqrt{2}\right)$, $\left(\frac{4}{\sqrt{6}},-\sqrt{2}\right)$, $\left(-\frac{4}{\sqrt{6}},-\sqrt{2}\right)$

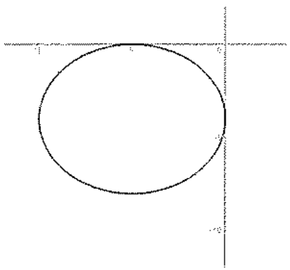
   Directrices: $y=\frac{6}{\sqrt{2}}$, $y=-\frac{6}{\sqrt{2}}$

   Eccentricity: $\frac{1}{\sqrt{3}}$



Section B5 (continued)

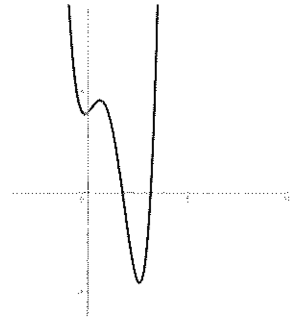
8. Center: $(-5,-4)$
   Vertices: $(0,-4)$, $(-10,-4)$
   Intercepts: $(-5,0)$, $(-5,-8)$
   Foci: $(8,-4)$, $(-2,-4)$
   ELRs:
   $\left(-8,-\frac{4}{5}\right)$, $\left(-8,-\frac{36}{5}\right)$, $\left(-2,-\frac{4}{5}\right)$, $\left(-2,-\frac{36}{5}\right)$
   Directrices: $x=\frac{-40}{3}$, $x=\frac{10}{3}$
   Eccentricity: $\frac{3}{5}$



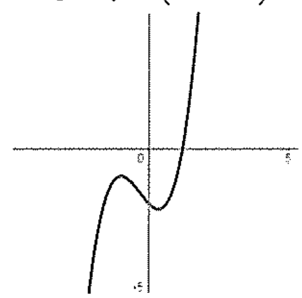
9. $\frac{(x+1)^2}{16}+\frac{(y-7)^2}{25}=1$
10. $\frac{(x-3)^2}{9}+\frac{(y+2)^2}{4}=1$
11. $\frac{x^2}{9}+\frac{y^2}{25}=1$
12. $\frac{(x-5)^2}{25}+\frac{y^2}{16}=1$
13. $\frac{(x-1)^2}{64}+\frac{(y+1)^2}{100}=1$

Section B6

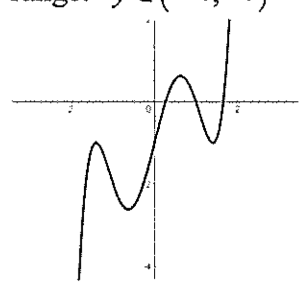
1. degree: 4
   $x$–intercept(s): (1.725, 0), (3.128, 0)
   y–intercept: (0, 4)
   minima: relative at (–0.164, 3.908);
   absolute at (2.574, –4.494)
   maxima: relative at (0.591, 4.586)
   domain: $x \in (-\infty, \infty)$
   range: $y \in [-4.494, \infty)$



2. degree: 3
   $x$–intercept(s): (1.206, 0)
   y–intercept: (0, –2)
   minima: relative at (0.333, –2.185)
   maxima: relative at (–1, –1)
   domain: $x \in (-\infty, \infty)$
   range: $y \in (-\infty, \infty)$

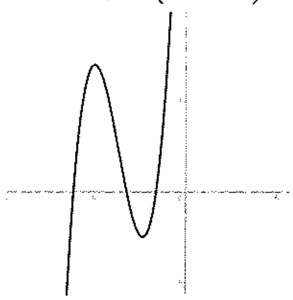


3. degree: 5
   $x$–intercept(s): (0.269, 0), (1, 0), (1.666, 0)
   y–intercept: (0, –1)
   minima: relatives at (–0.632, –2.619), (1.414, –1)
   maxima: relatives at (–1.414, –1), (0.632, 0.619)
   domain: $x \in (-\infty, \infty)$
   range: $y \in (-\infty, \infty)$

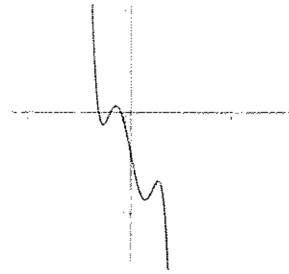


Section B6 (continued)

4. degree: 3
   $x$–intercept(s): (–6.164, 0), (–3.227, 0), (–1.609, 0)
   y–intercept: (0, 32)
   minima: relative at (–2.333, –2.481)
   maxima: relative at (–5, 7)
   domain: $x \in (-\infty, \infty)$
   range: $y \in (-\infty, \infty)$



5. degree: 5
   $x$–intercept(s): (–1.585, 0), (–1, 0), (–0.487, 0)
   y–intercept: (0, –2)
   minima: relatives at (–1.365, –0.609), (0.733, –4.301)
   maxima: relatives at (–0.733, 0.301), (1.365, –3.391)
   domain: $x \in (-\infty, \infty)$
   range: $y \in (-\infty, \infty)$



6. degree: 4
   $x$–intercept(s): (–1.325, 0), (0, 0)
   y–intercept: (0, 0)
   minima: absolute at (–0.885, –1.055)
   maxima: none
   domain: $x \in (-\infty, \infty)$
   range: $y \in [-1.054, \infty)$