How the secant line became the tangent line

A story of two points coming together

Secant Lines

- Draw a line through these two points...

Hi, I'm Mr. Murphy. You might wonder what I'm doing up here at the top of this diving board. Well, the very industrious Chiarra and Lauren have convinced me to help out with the school fundraiser by jumping off this high diving board and since a lot of students bought tickets, how can I say no?

This is also an opportunity for our class to learn a few basics about Pre-Calculus. Since this diving board is 196 feet above the water tank and since many of you recall from Physics how gravity affects falling objects, we can easily determine the equation for my height above the water tank at any particular instant during my dive.

Why? Because I won't dive until you all do two things:

1. Find out how long will it take me to reach the water
2. Use secant and tangent lines to determine how fast will I be going when I hit the water?


Now recall that the first thing we need to do is...
1. Diagram the event.
Let's do that now, shall we?

The fall as stated before was 196 feet.

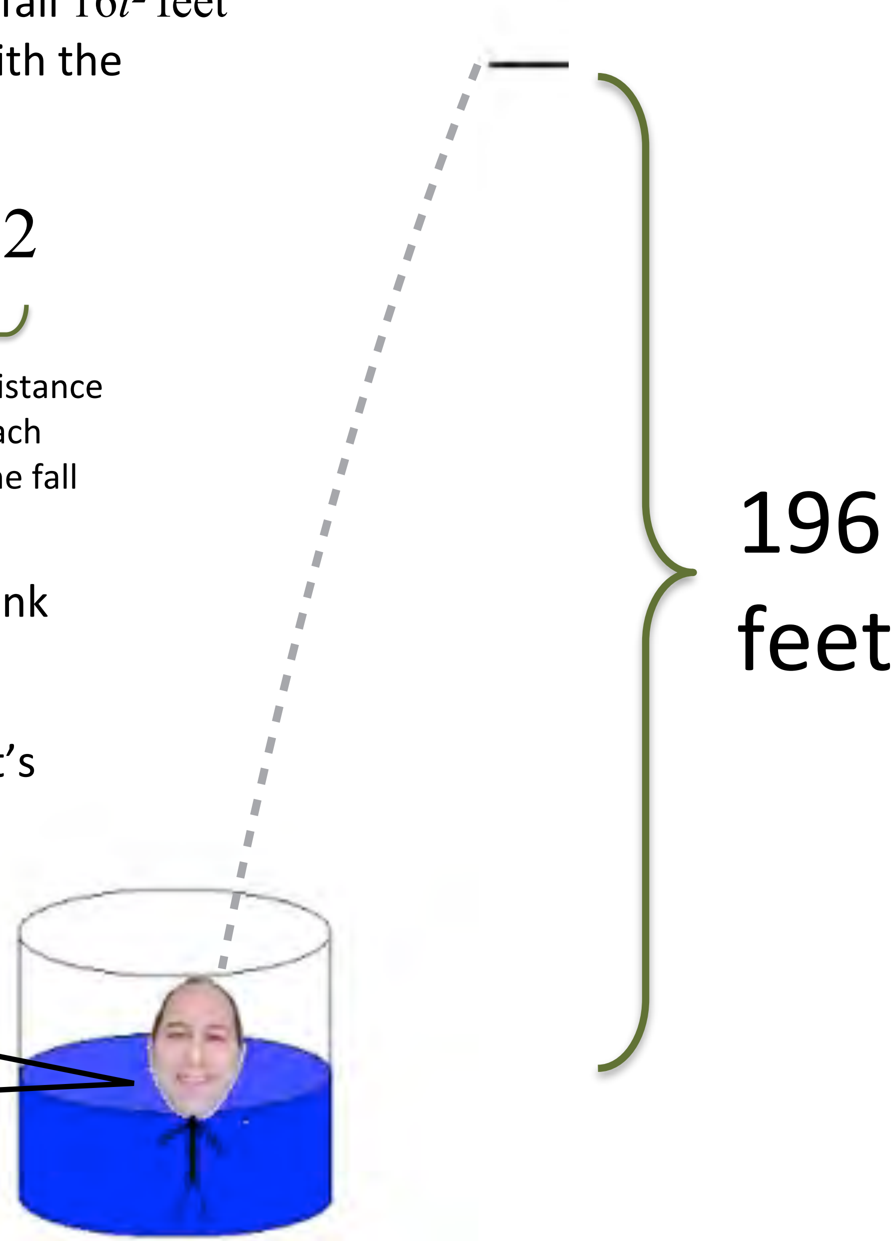
Earth's gravity causes Mr. Murphy to fall $16t^2$ feet where t is given in seconds starting with the beginning of the dive.

$$s(t) = \underbrace{196}_{\text{Mr. Murphy's height at the beginning}} \underbrace{- 16t^2}_{\text{minus the distance he covers each second of the fall}}$$

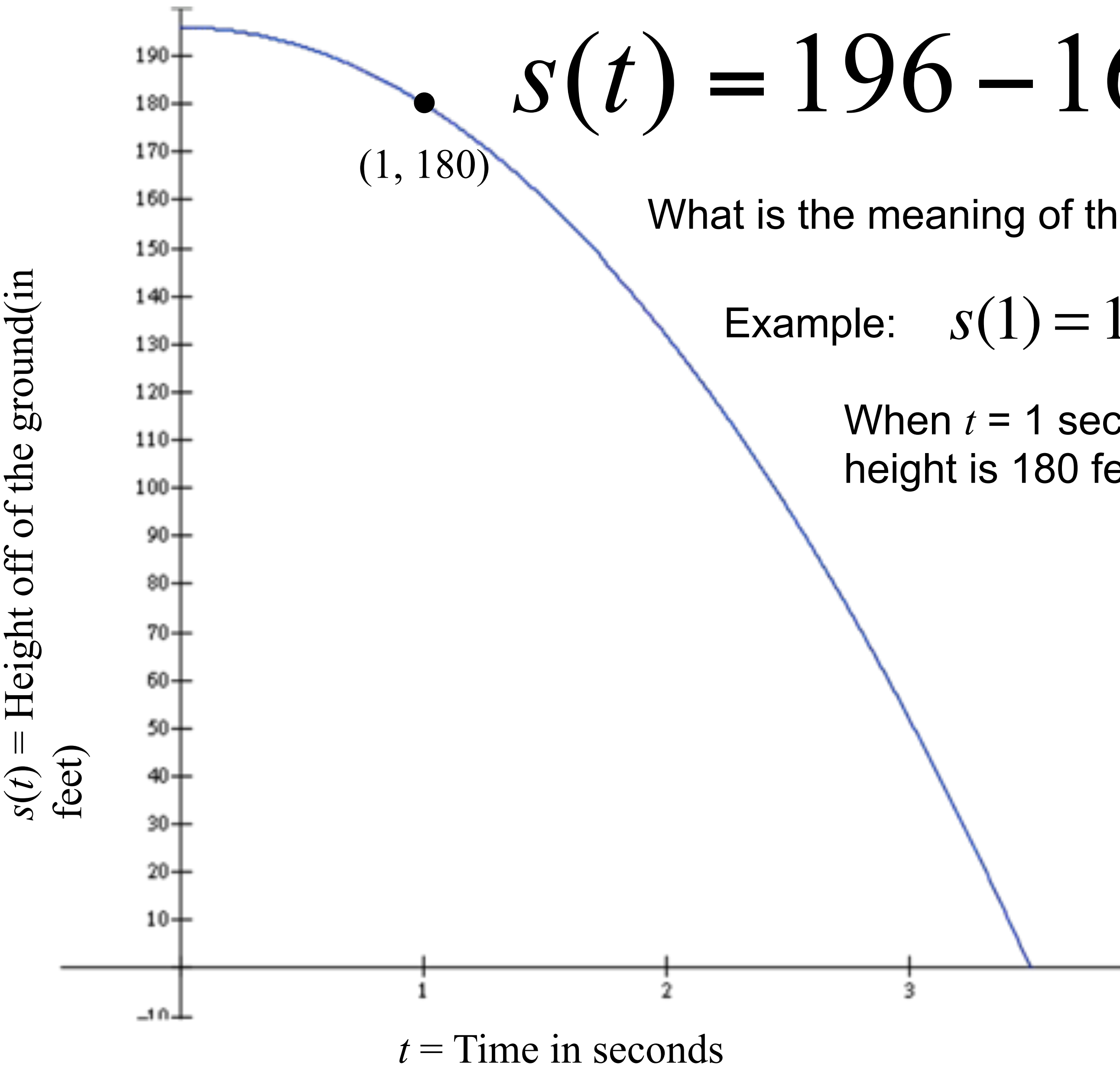
$s(t)$: Mr. Murphy's height above the water tank at time t

Since our story today is about slopes, let's graph this equation.

Which bring us to...
2. Write an equation. Given the info, we now have



$$s(t) = 196 - 16t^2$$



What is the meaning of this equation and graph?

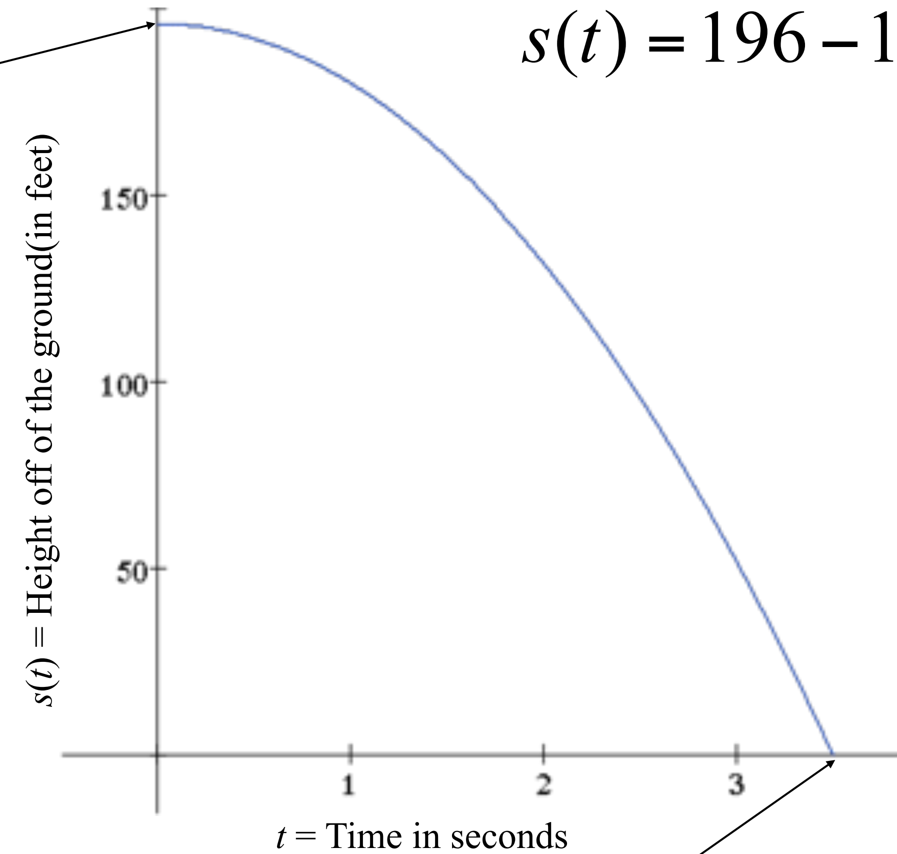
Example: $s(1) = 196 - 16(1)^2$

When t = 1 second, Mr. Murphy's height is 180 feet

$$s(t) = 196 - 16t^2$$

Just to be sure, what are the coordinates for this point?

(0, 196) because at time = 0, Mr. Murphy is at the top of the 196 foot high platform.



Now that we have everything set up, they must answer my first question: How long will it take me to reach the water?

In other words, when is my height above the tank equal to 0?

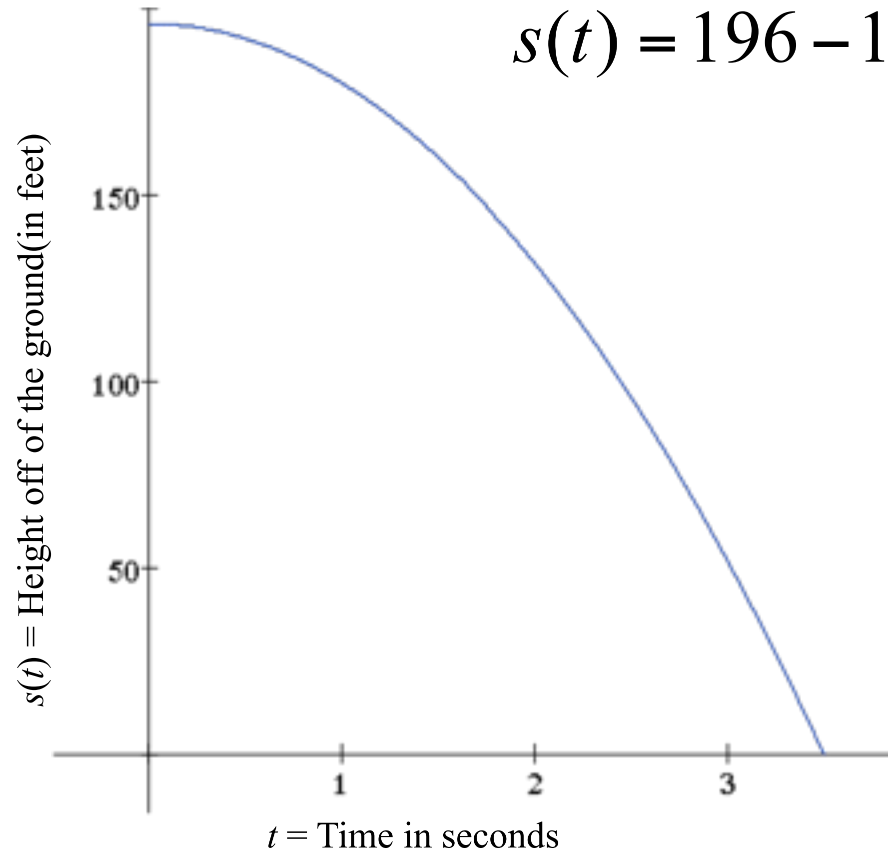
$$s(t) = 196 - 16t^2 = 0$$

$t = 3.5$ seconds

$$s(t) = 196 - 16t^2$$

Now onto the second question. But wait, what does this whole idea of the secant line meeting the tangent line have to do with Mr. Murphy's velocity?

Let's start by finding average velocity which is very straightforward.



What is Mr. Murphy's *average velocity* during his 3.5 second plunge?

$$v_{avg} = \frac{s(3.5) - s(0)\ \ \textit{feet}}{3.5 - 0\ \ \text{sec}}$$

$$v_{avg} = \frac{0 - 196\ \ \textit{feet}}{3.5 - 0\ \ \text{sec}}$$

Quick Physics Review: Why is the velocity negative?

Because the motion is downward.

$$v_{avg} = \frac{-196\ \ \textit{feet}}{3.5\ \ \text{sec}} = -56 \text{ feet/sec}$$

Since you are all experts at algebra…

$$v_{avg} = \frac{s(3.5) - s(0) \quad feet}{3.5 - 0 \quad \text{sec}}$$

Doesn't this quotient look familiar?

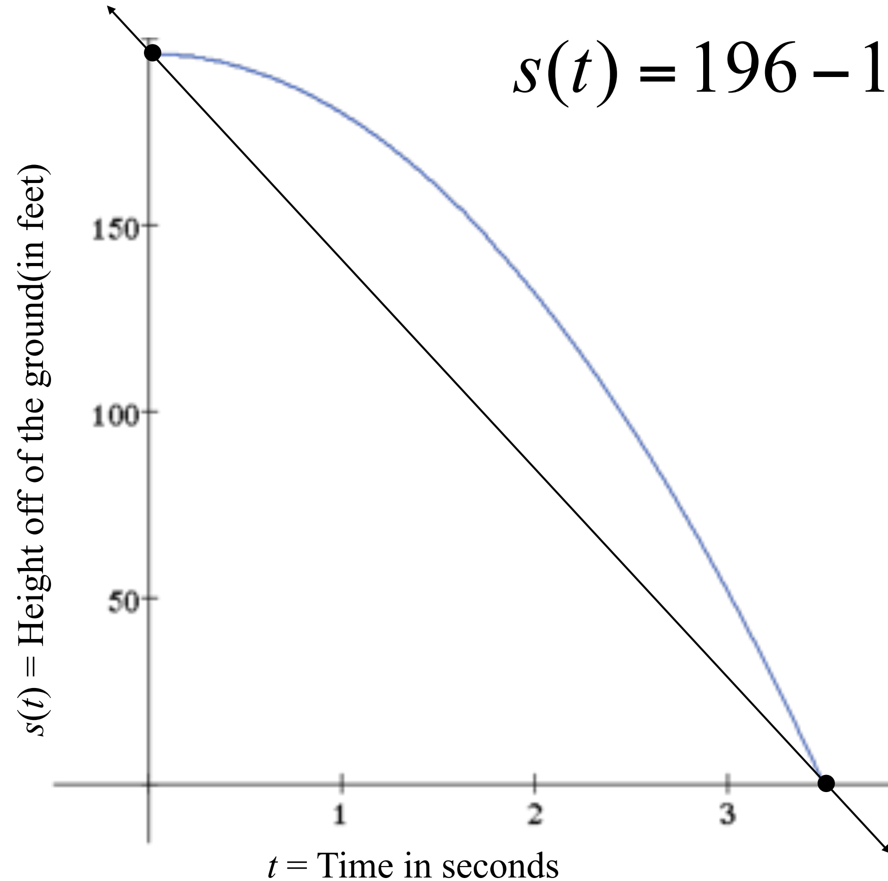
$$\frac{s(3.5) - s(0)}{3.5 - 0} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$v_{avg} = \frac{-196 \quad feet}{3.5 \quad \text{sec}}$$

$$= -56 \text{ feet/sec}$$

…can be shown graphically to be…

$$s(t) = 196 - 16t^2$$



The average velocity is also the slope of the secant line through these two points.

m_{sec} from time 0 to time 3.5

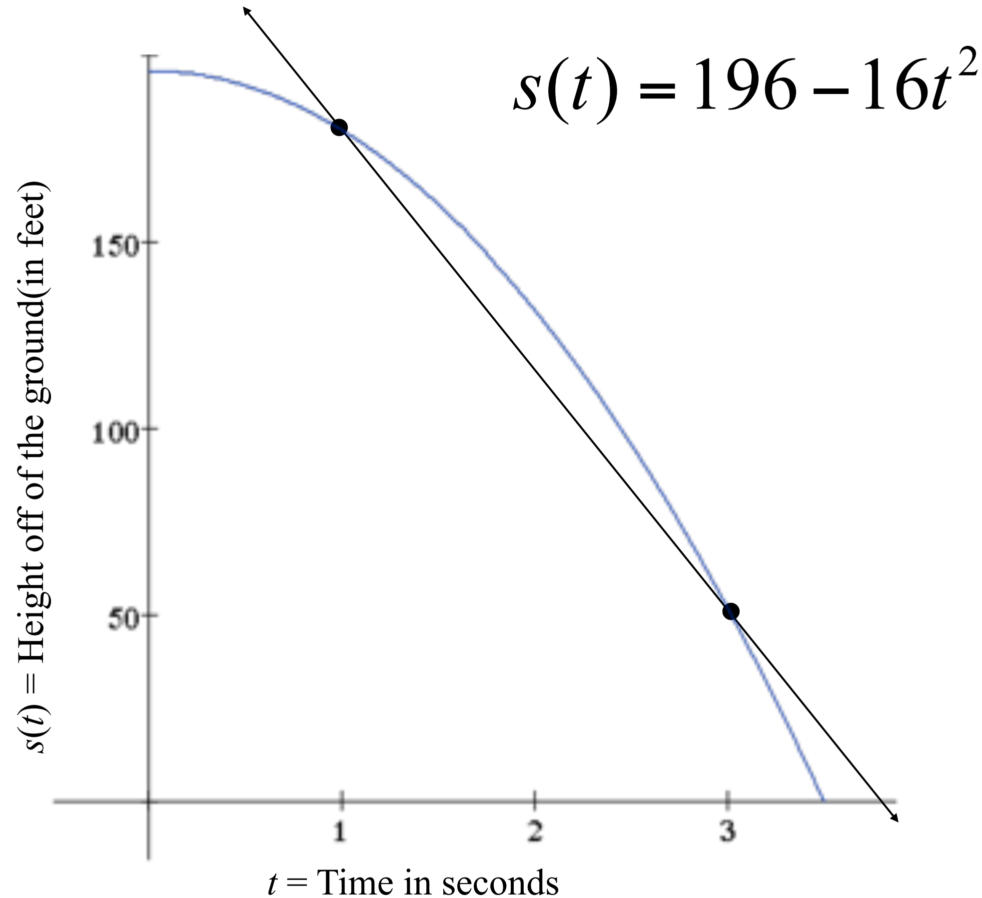
Just for good measure, find Mr. Murphy's average velocity between 1 and 3 seconds.

$$v_{avg} = \frac{s(3) - s(1) \quad feet}{3 - 1 \quad \text{sec}}$$

$$v_{avg} = \frac{52 - 180 \quad feet}{3 - 1 \quad \text{sec}}$$

$$v_{avg} = \frac{-128 \quad feet}{2 \quad \text{sec}}$$

$$= -64 \text{ feet/sec}$$



So now there's a connection with the secant line.

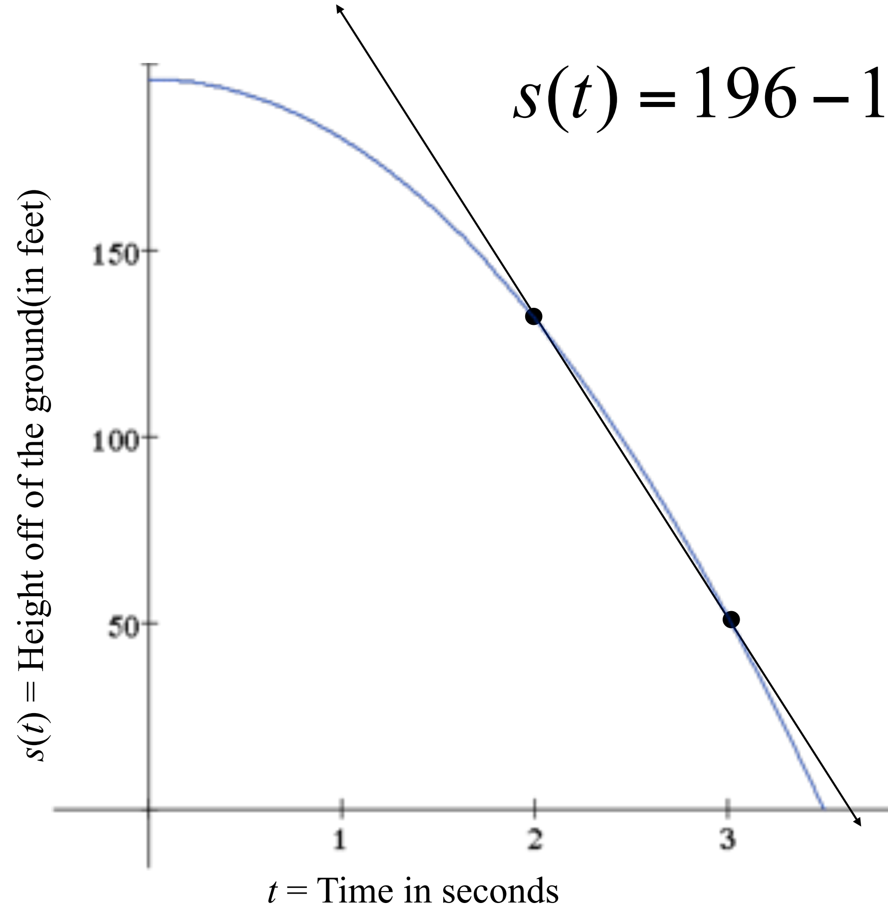
m_{sec} from time 1 to time 3

But what about the tangent line and the exact velocity?

Approximate Mr. Murphy's *instantaneous* (exact) velocity at 3 seconds.

We can draw a secant line close to 3. we'll start with 2 seconds.

$$s(t) = 196 - 16t^2$$



$$v_{avg} = \frac{s(3) - s(2) \quad feet}{3 - 2 \quad \text{sec}}$$

$$v_{avg} = \frac{52 - 132 \quad feet}{3 - 2 \quad \text{sec}}$$

$$v_{avg} = \frac{-80 \quad feet}{1 \quad \text{sec}}$$

…which is close to the exact velocity at 3 seconds.

$$= -80 \text{ feet/sec}$$ → m_{sec} from time 2 to time 3

Approximate Mr. Murphy's *instantaneous* (exact) velocity at 3 seconds.

We can even try a secant line through 2.5 and 3.

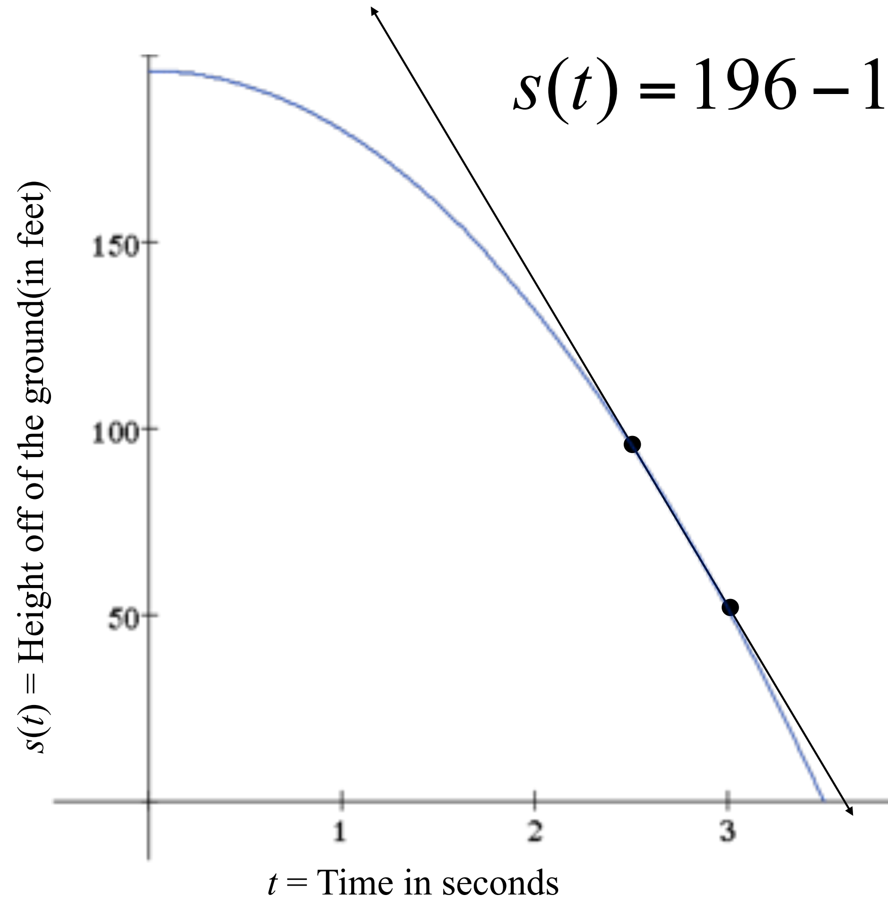
$$v_{avg} = \frac{s(3) - s(2.5) \quad feet}{3 - 2.5 \quad \text{sec}}$$

$$v_{avg} = \frac{52 - 96 \quad feet}{3 - 2.5 \quad \text{sec}}$$

$$v_{avg} = \frac{-44 \quad feet}{0.5 \quad \text{sec}}$$

$= -88$ feet/sec → m_{sec} from time 2.5 to time 3

$$s(t) = 196 - 16t^2$$



…which is even closer to the exact velocity at 3 seconds.

Approximate Mr. Murphy's *instantaneous* (exact) velocity at 3 seconds.

We can even try a secant line through 2.9 and 3.

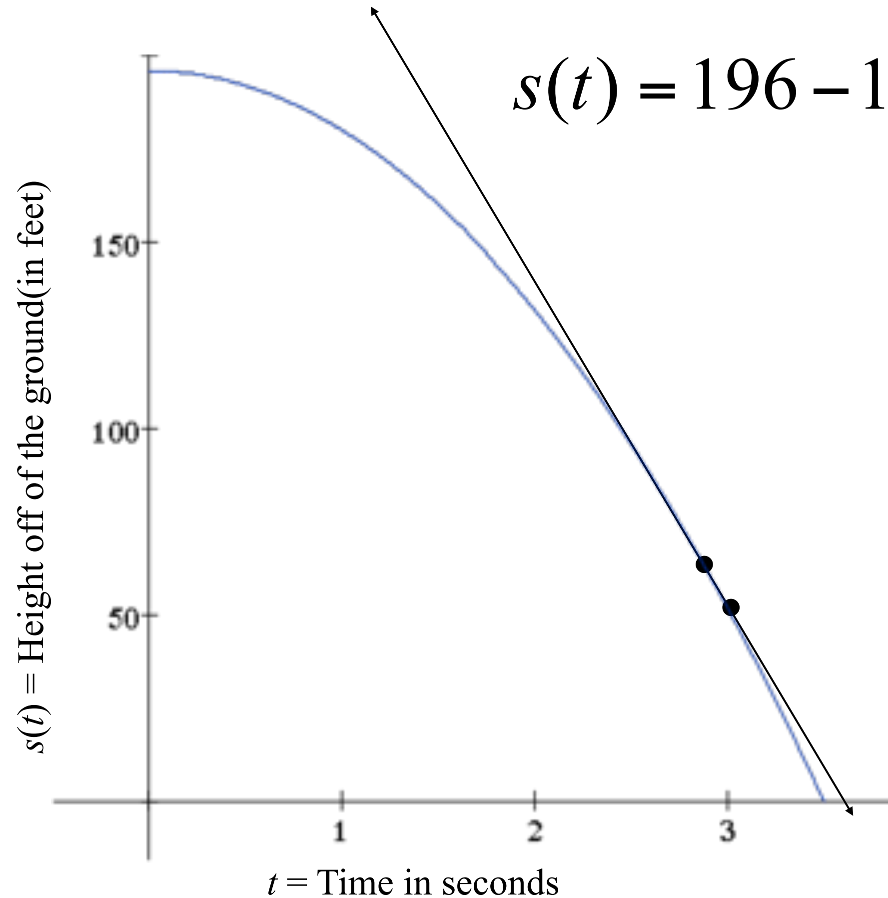
$$v_{avg} = \frac{s(3) - s(2.9)\ feet}{3 - 2.9\text{ sec}}$$

$$v_{avg} = \frac{52 - 61.44\ feet}{3 - 2.9\text{ sec}}$$

$$v_{avg} = \frac{-9.44\ feet}{0.1\text{ sec}}$$

$$= -94.4\text{ feet/sec}$$

$$s(t) = 196 - 16t^2$$



…which is even closer to the exact velocity at 3 seconds.

m_{sec} from time 2.9 to time 3

Approximate Mr. Murphy's *instantaneous* (exact) velocity at 3 seconds.

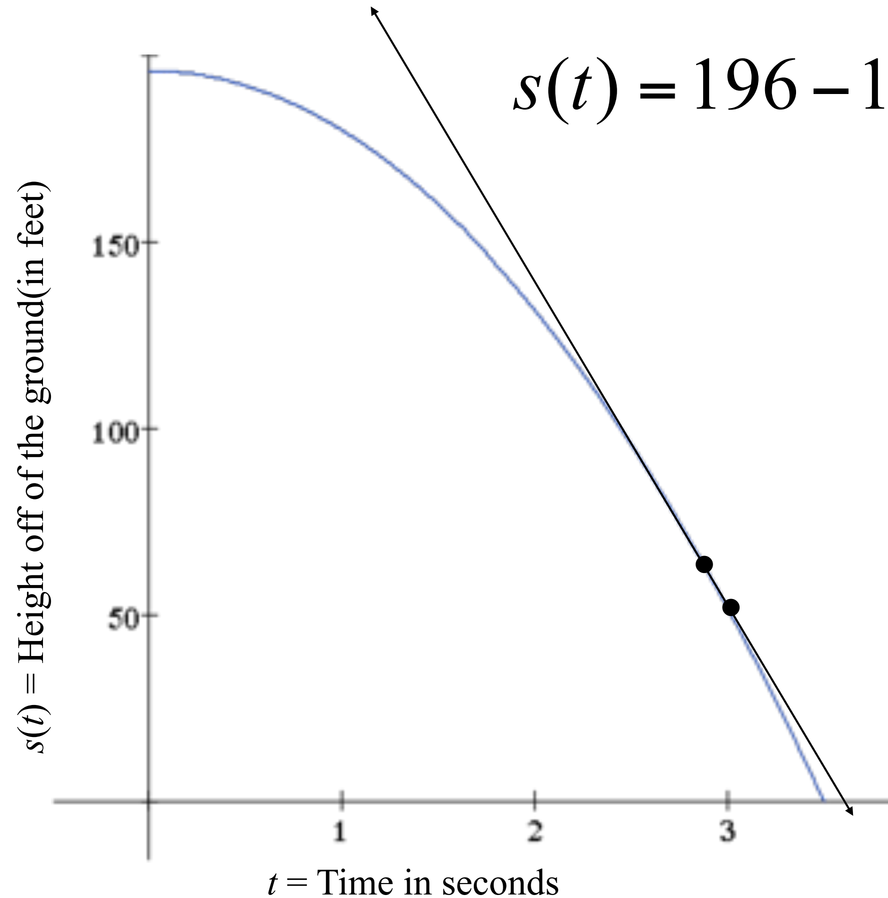
What if we put the points right on top of each other?

$$v_{avg} = \frac{s(3) - s(3)\ feet}{3 - 3\,\text{sec}}$$

$$v_{avg} = \frac{0\ feet}{0\ \text{sec}}$$

This is neither 0 nor "undefined" as you learned with vertical slopes in Algebra.

$$s(t) = 196 - 16t^2$$



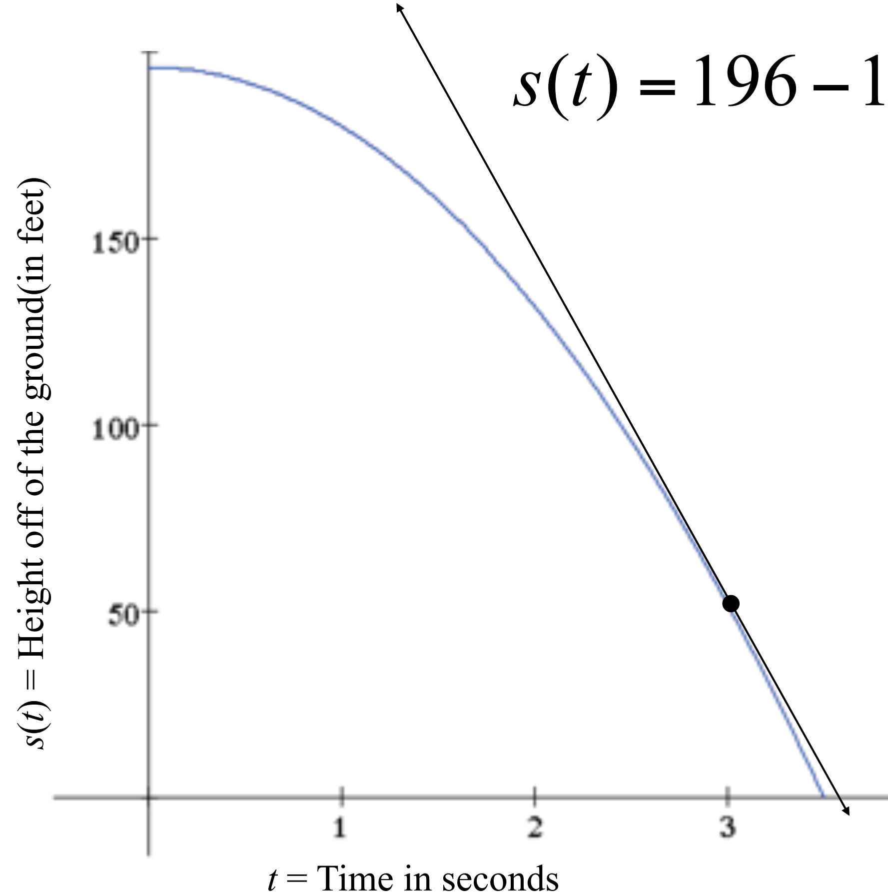
So what can we do about this?

There is something...

There is a very important concept in Pre-Calculus called a *Limit*

You won't need to know much about it but we are going to do a shorthand version of it here.

$$s(t) = 196 - 16t^2$$



Whenever we have a result of 0/0 as we just did, we can look for a way to simplify the fraction like this:

$$v = \lim_{t \to 3} \frac{s(t) - s(3)}{t - 3} \frac{\text{feet}}{\text{sec}}$$

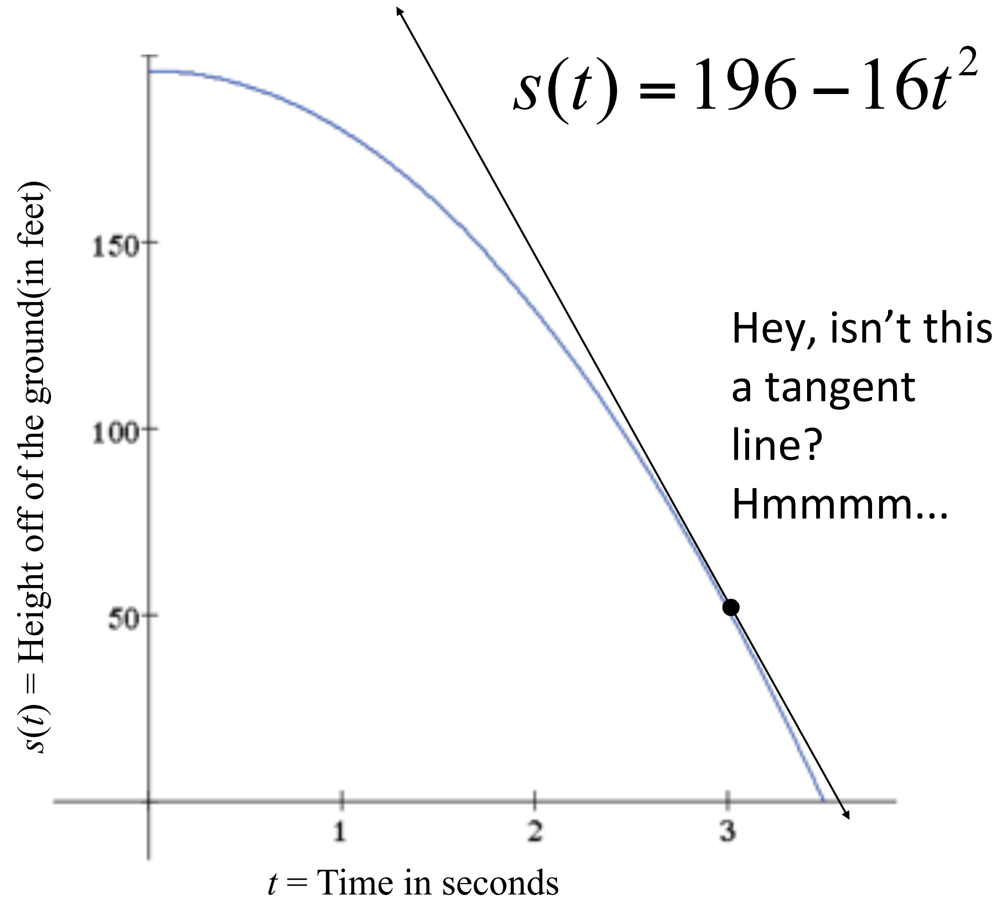
Don't panic at this. It just means "what happens when *t* gets closer and closer to 3?" We have a simple algebraic technique for dealing with this.

$$v = \lim_{t \to 3} \frac{\overbrace{196 - 16t^2}^{s(t)} - \overbrace{(196 - 16 * 3^2)}^{s(3)}}{t - 3} \frac{\text{feet}}{\text{sec}}$$

When we take a limit like this, we first notice that plugging in the number (in this case $t = 3$) would give us $0/0$

When that happens, we look for a way to simplify the fraction by factoring and canceling terms.

Let's do it with this limit.



$$v = \lim_{t \to 3} \frac{s(t) - s(3) \quad feet}{t - 3 \quad \text{sec}}$$

$$v = \lim_{t \to 3} \frac{196 - 16t^2 - (196 - 16 * 3^2) \quad feet}{t - 3 \quad \text{sec}}$$

$$v = \lim_{t \to 3} \frac{196 - 16t^2 - 52 \quad feet}{t - 3 \quad \text{sec}} = \lim_{t \to 3} \frac{144 - 16t^2}{t - 3} = \lim_{t \to 3} \frac{-16(t^2 - 9)}{t - 3}$$

When we take a limit like this, we first notice that plugging in the number (in this case $t = 3$) would give us $0/0$

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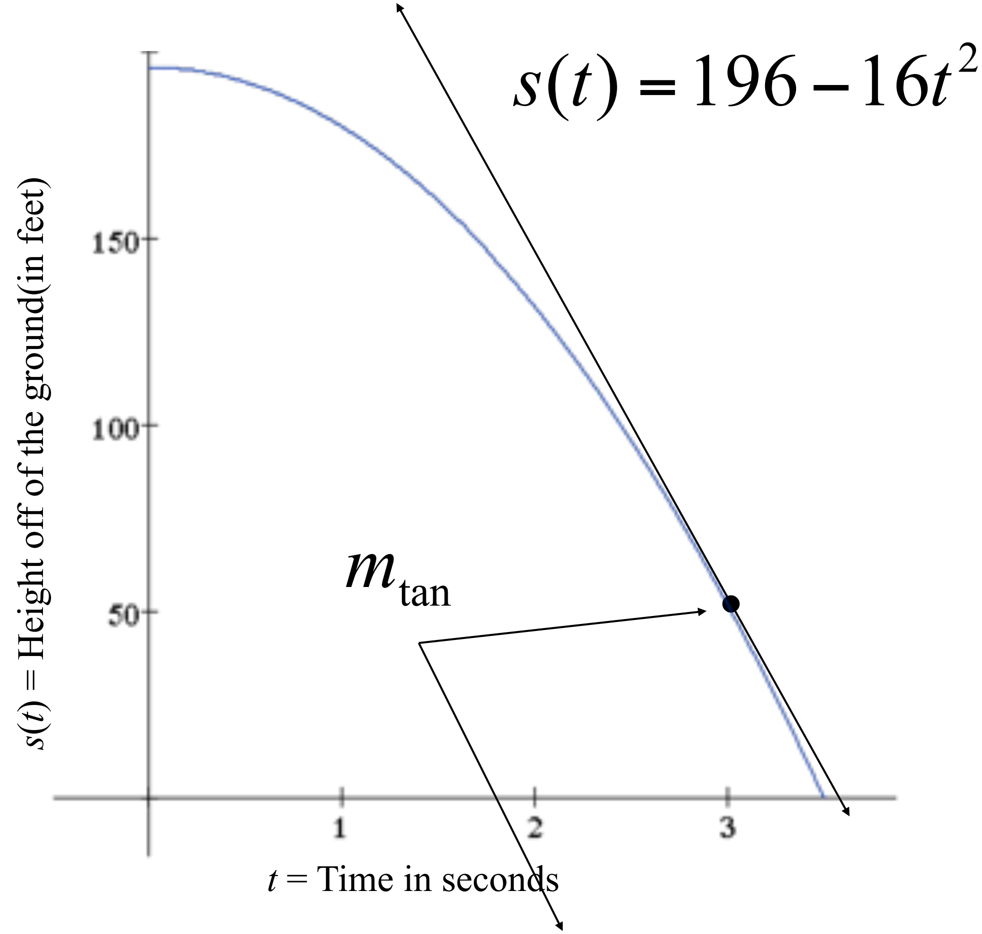
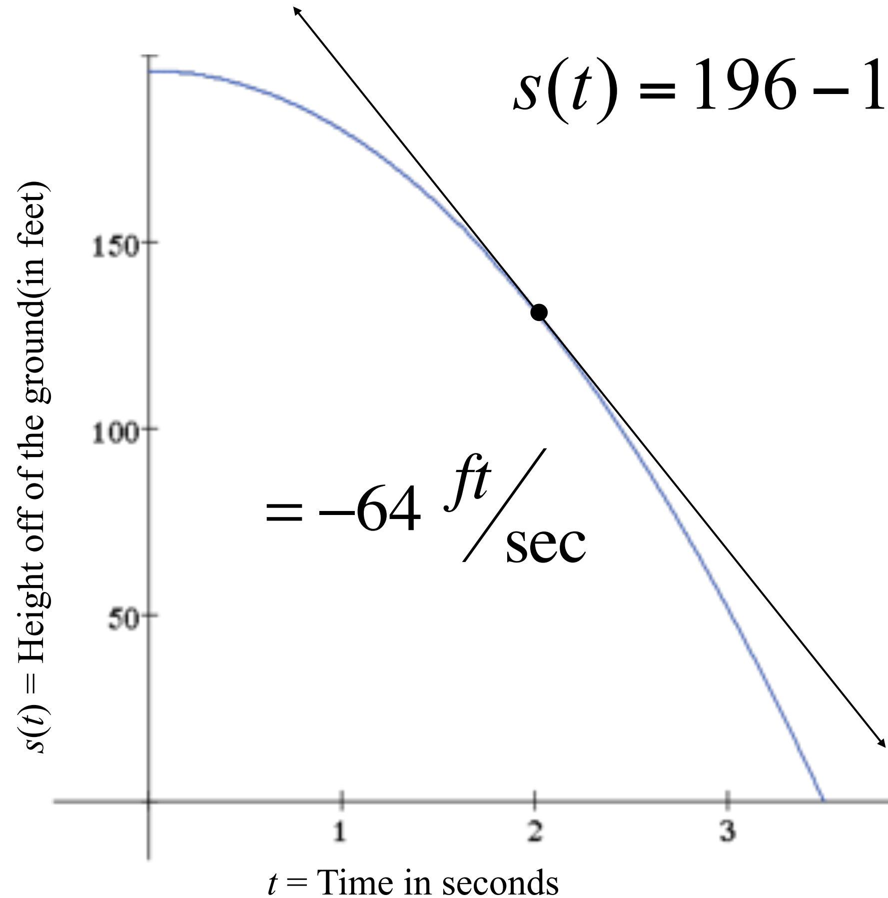


$$v = \lim_{t \to 3} \frac{-16(t^2 - 9)}{t - 3}$$

$$v = \lim_{t \to 3} \frac{-16(t-3)(t+3)}{t-3} = \lim_{t \to 3} -16(t+3) = -96\ feet/\text{sec}$$

$$s(t) = 196 - 16t^2$$

So since we know how long Mr. Murphy was in the air, let's find his *instantaneous* (exact) velocity at 2 seconds.

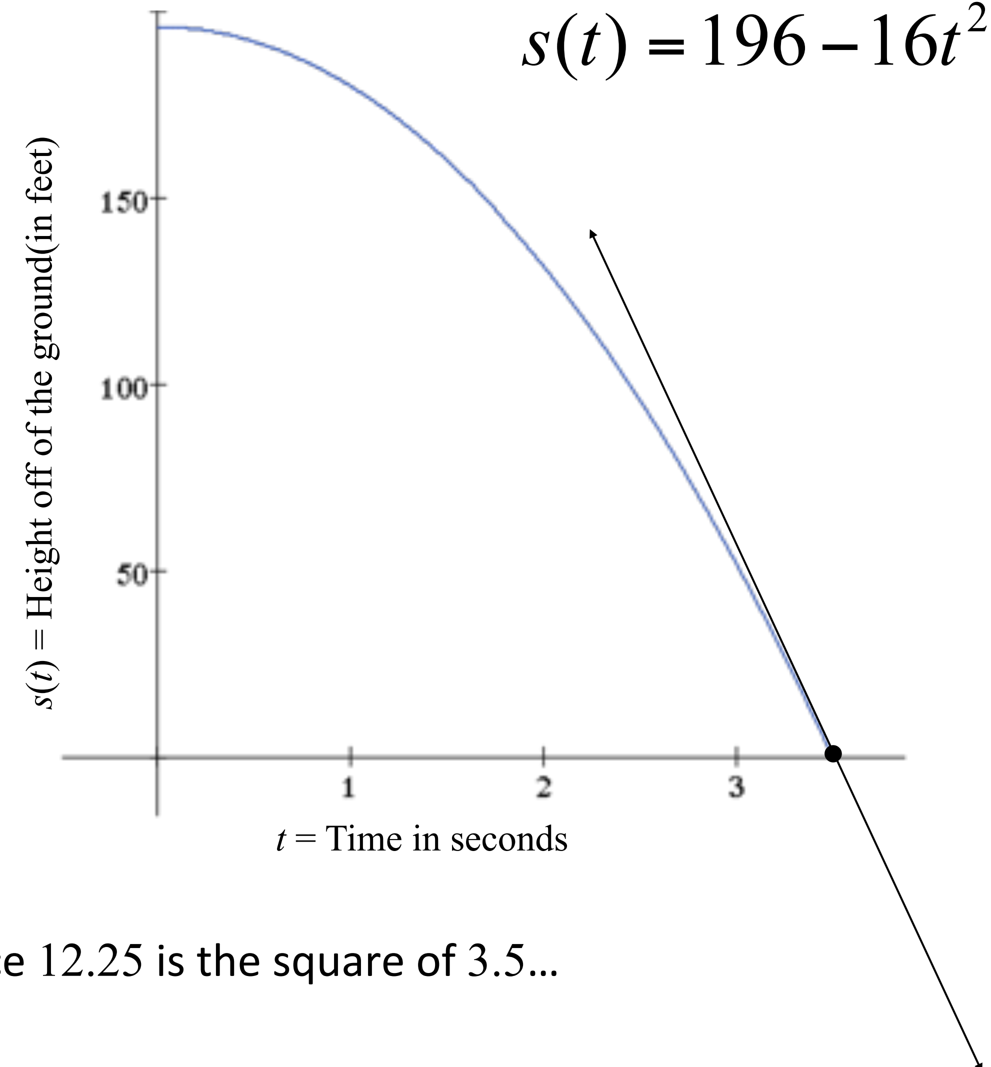


$$v = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2}$$

$$v = \lim_{t \to 2} \frac{196 - 16t^2 - (196 - 16 \cdot 2^2)}{t - 2} = \lim_{t \to 2} \frac{196 - 16t^2 - 132}{t - 2}$$

$$= \lim_{t \to 2} \frac{64 - 16t^2}{t - 2} = \lim_{t \to 2} \frac{-16(t^2 - 4)}{t - 2} = \lim_{t \to 2} \frac{-16(t - 2)(t + 2)}{t - 2}$$

$$s(t) = 196 - 16t^2$$



Now let's find his ***instantaneous*** (exact) velocity at 3.5 seconds.

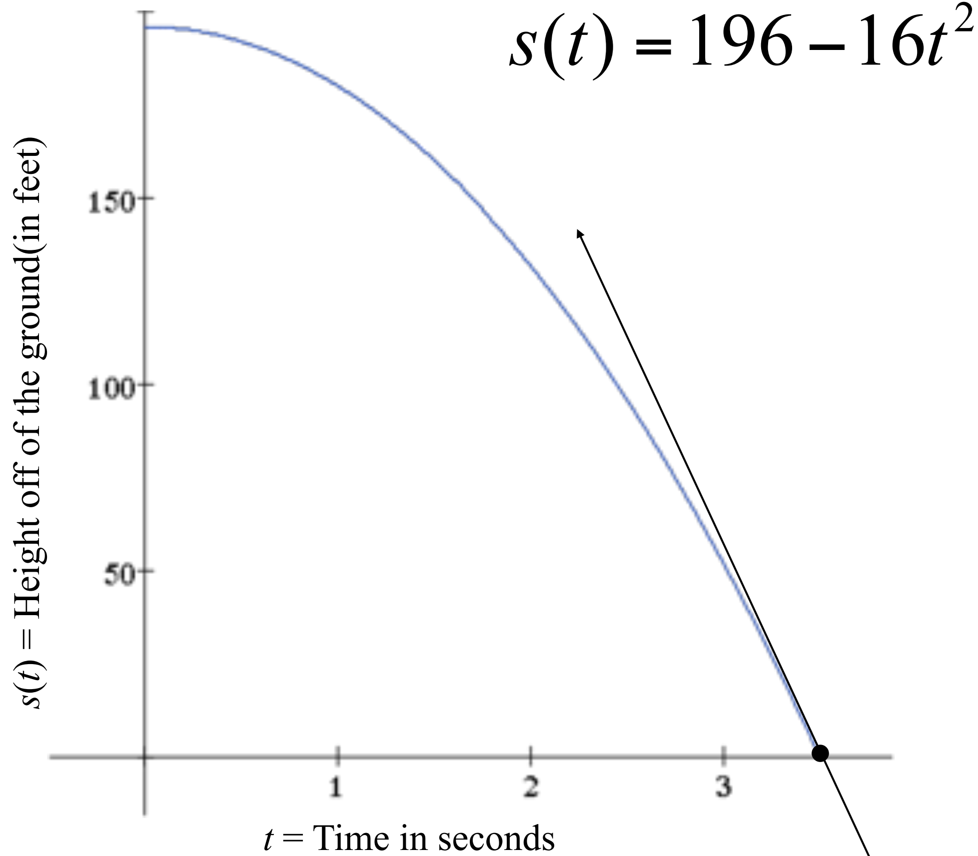
$$v = \lim_{t \to 3.5} \frac{s(t) - s(3.5)}{t - 3.5} \frac{\textit{feet}}{\text{sec}}$$

$$\lim_{t \to 3.5} \frac{-16(t^2 - 12.25)}{t - 3.5}$$

Since 12.25 is the square of 3.5...

$$\lim_{t \to 3.5} \frac{-16\cancel{(t - 3.5)}(t + 3.5)}{\cancel{t - 3.5}} = -112 \text{ feet/sec}$$

$$s(t) = 196 - 16t^2$$



So does this mean that we will have to do all of this factoring and canceling every time?

Actually, no. There is a shorter way but you need to see the long way first for reasons we won't go into here.

Also, the method you're about to see only applies to power terms in functions. In fact, it is a rule called...

THE POWER RULE

Given the function x^n the slope of the tangent line is nx^{n-1}

when you have a constant, it drops to become 0 as you will see...

Finding the slope of the tangent line is also called *taking the derivative* of the function.

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Here's how it works:

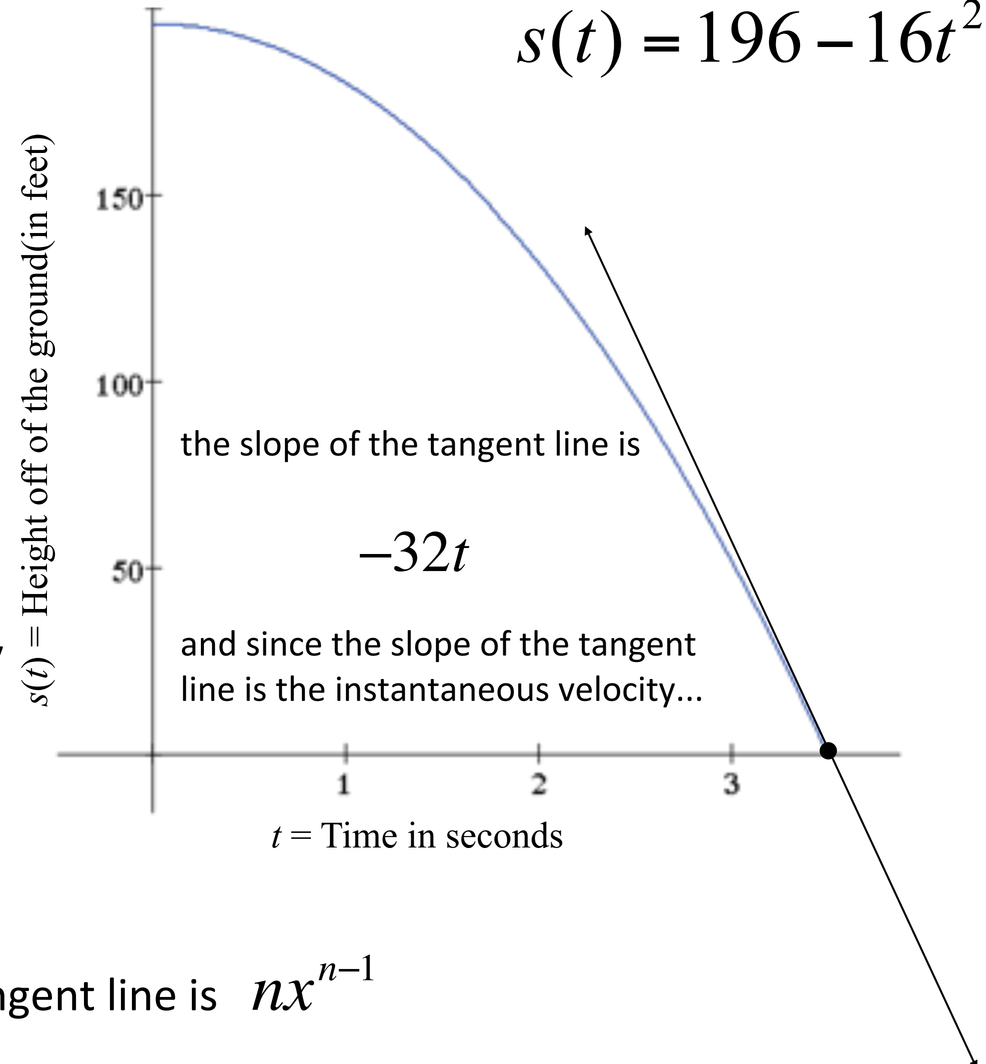
Given a function $f(x) = x^3 - x^2 + 5$ its derivative is $f(x) = 3x^2 - 2x + 0$

Given a function $f(x) = 2x^3 - 5x^2 + 5x - 3$ its derivative is $f(x) = 6x^2 - 10x + 5$

Given a function $f(x) = 3x^4 + 7x^2 + 11$ its derivative is $f(x) = 12x^3 + 14x$

Now let's apply the power rule to our high dive problem

$$s(t) = 196 - 16t^2$$



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THE POWER RULE

Given the function x^n the slope of the tangent line is nx^{n-1}

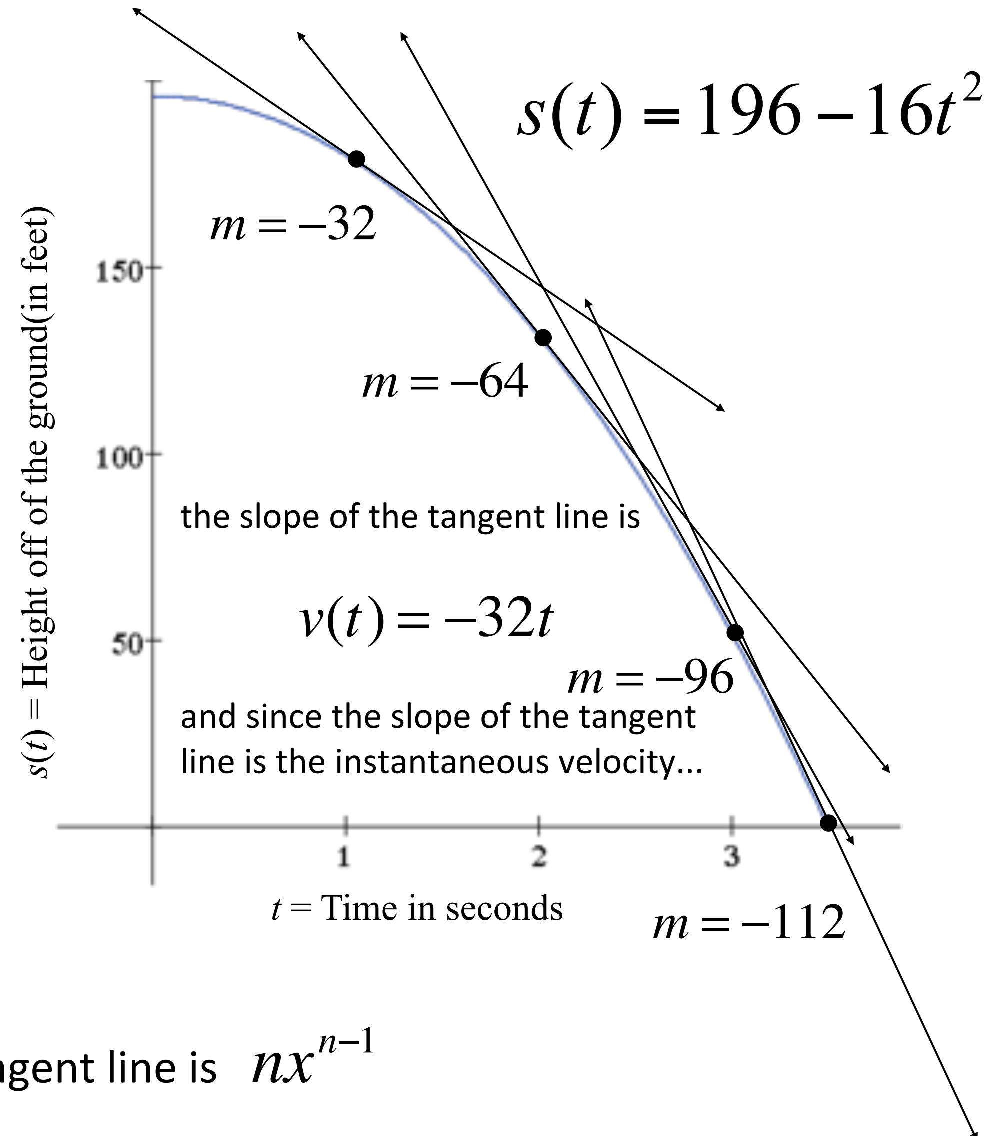
Given the function $s(t) = 196 - 16t^2$

when you have a constant, it drops to become 0

the slope of the tangent line (*the derivative*) is $s'(t) = v(t) = 0 - 2 \cdot 16t^{2-1}$

nx^{n-1} using the Power Rule with $n = 2$

t	$s(t)=196-16t^2$	$v(t)=-32t$
1	180 feet	-32 ft/sec
2	132 feet	-64 ft/sec
3	52 feet	-96 ft/sec
3.5	0	-112 ft/sec



THE POWER RULE

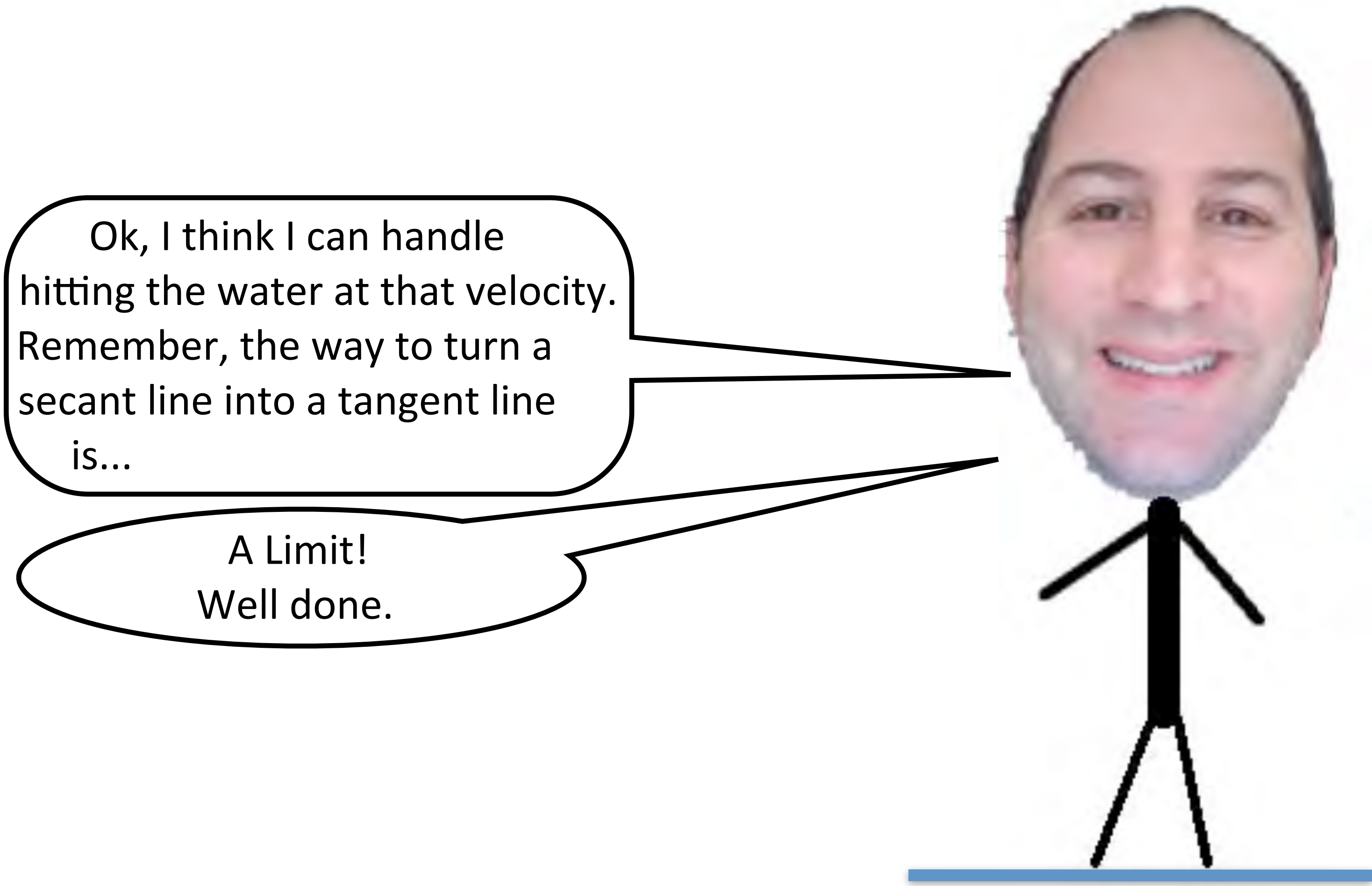
Given the function x^n the slope of the tangent line is nx^{n-1}

Given the function $s(t)=196-16t^2$

when you have a constant, it drops to become 0 as you will see...

the slope of the tangent line (*the derivative*) is $s'(t)=v(t)=0-2\cdot 16t^{2-1}$

nx^{n-1} using the Power Rule with $n=2$


Ok, I think I can handle hitting the water at that velocity. Remember, the way to turn a secant line into a tangent line is...
A Limit! Well done.


And we found the Power Rule to be an easier way to find the slope of a tangent line.
Enough talk! Come on! Dive!
The only way you come down is diving!
Yeah! Hurry up and dive!