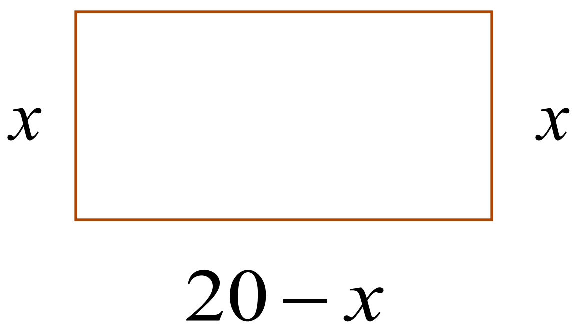
3-3 Optimization

Finding Optimum Values

A Classic Problem

You have 40 feet of fence to enclose a rectangular garden. What is the maximum area that you can enclose?



$$w = x$$

$$l = \frac{40 - 2x}{2} = 20 - x$$

$$\boxed{\begin{aligned} w &= 10 \text{ ft} \\ l &= 10 \text{ ft} \end{aligned}}$$

$$A(x) = x(20 - x)$$

Consider $A(x)$ to be the area formula as a function of x. What is the domain?

$$0 < x < 20 \; feet$$

How do we find the maximum value of $A(x)$?

$$A(x) = 20x - x^2$$

$$A'(x) = 20 - 2x \longleftarrow$$ Take the *derivative!*

Set it to zero and solve for x.

$$A'(x) = 20 - 2x = 0 \quad \longrightarrow \quad x = 10$$

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To solve problems involving optimization:

- **Draw a picture!!!** These problems really require that you get a good look at what you are trying to find.

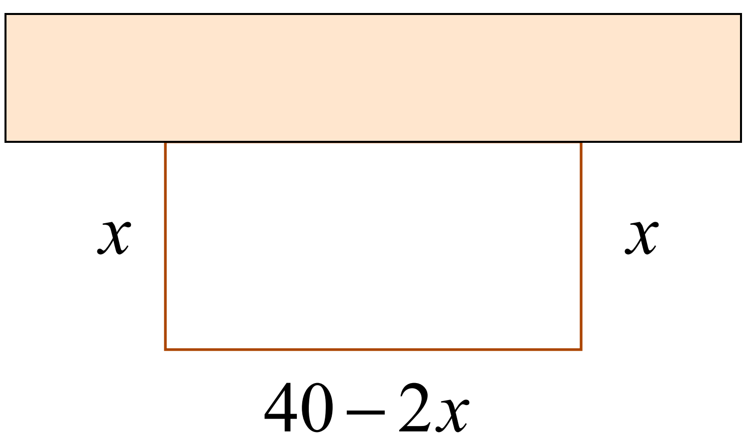
- Label your dimensions in terms of one variable (usually x)

- Find a function in terms of x to represent what maximum/minimum value you are trying to find. Be sure to define the domain for x.

- Take the derivative, set it equal to zero and find the critical points.

- Check the critical points and don't forget the endpoints.

You have 40 feet of fence to enclose a rectangular garden along the side of a barn. What is the maximum area that you can enclose?



$$A = x(40-2x)$$

$$A = 40x - 2x^2$$

$$A' = 40 - 4x$$

$$0 = 40 - 4x$$

$$4x = 40$$

$$x = 10$$

There must be a local maximum here, since the endpoints are minimums.

$$w = x$$

$$l = 40 - 2x$$

$$\boxed{\begin{aligned} w &= 10 \text{ ft} \\ l &= 20 \text{ ft} \end{aligned}}$$

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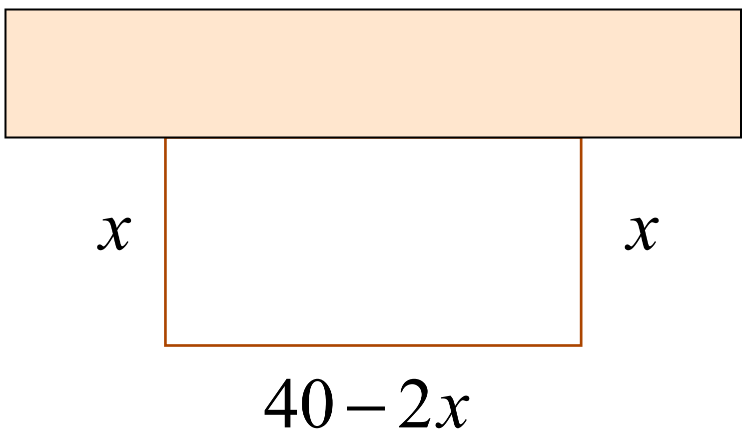


$$A = x(40-2x)$$

$$A = 40x - 2x^2$$

$$A' = 40 - 4x$$

$$0 = 40 - 4x$$

$$4x = 40$$

$$x = 10$$

$$A = 10(40 - 2\cdot 10)$$

$$A = 10(20)$$

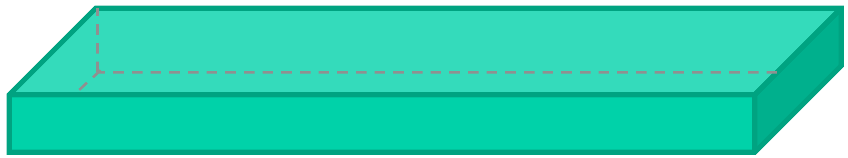
$$\boxed{A = 200 \text{ ft}^2}$$

$$w = x$$

$$l = 40 - 2x$$

$$\boxed{\begin{aligned} w &= 10 \text{ ft} \\ l &= 20 \text{ ft} \end{aligned}}$$

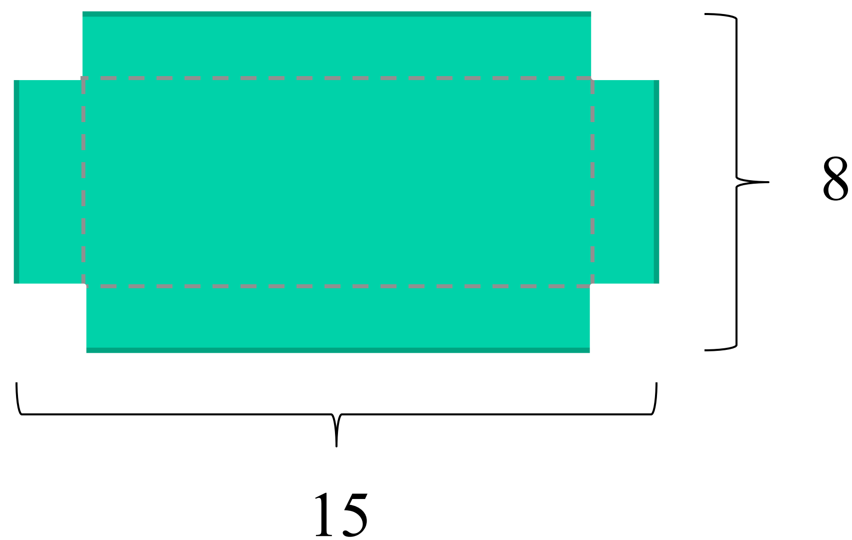
You are making an open top rectangular box



Find the dimensions of such a box with the largest volume.

from an 8 by 15 inch piece of cardboard

by cutting square pieces off of each corner and folding up the sides



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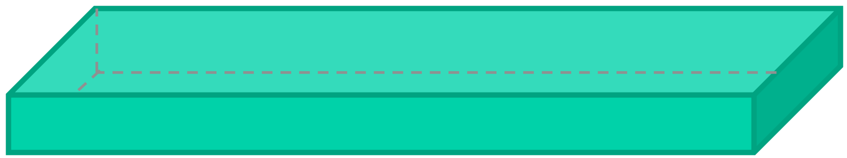


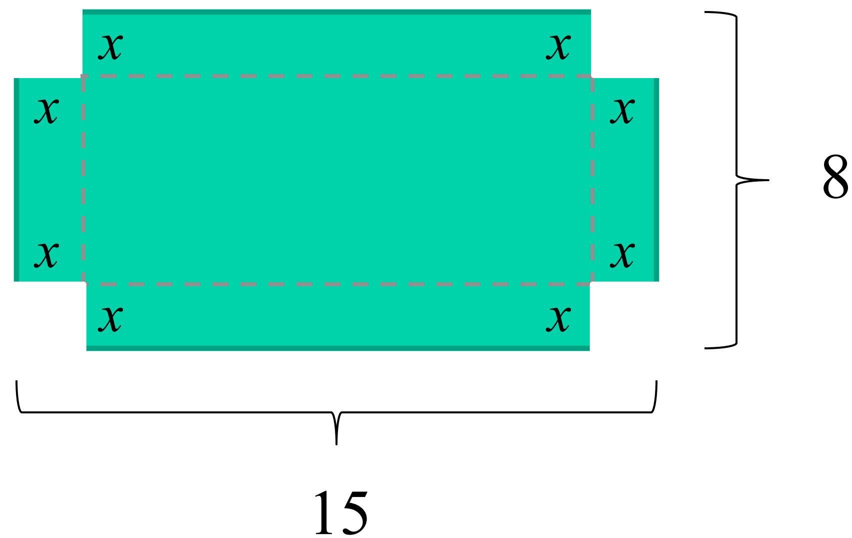
Find the dimensions of such a box with the largest volume.

from an 8 by 15 inch piece of cardboard

Each square that we cut off would have sides equal to x

$$0 < x < 4$$ Why?

Do we have a limited domain for x values?

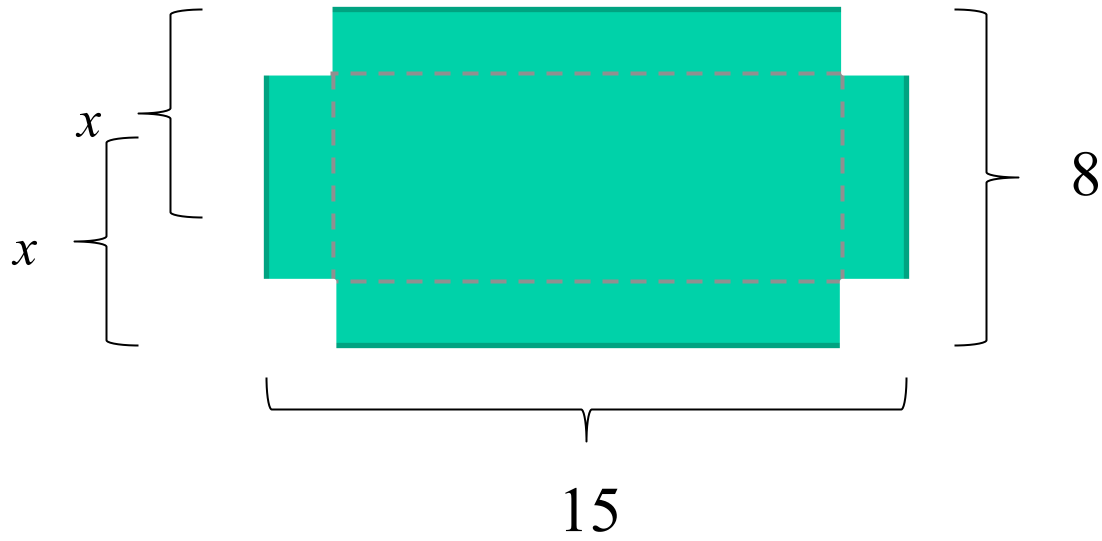


Each square that we cut off would have sides equal to x

$$0 < x < 4$$

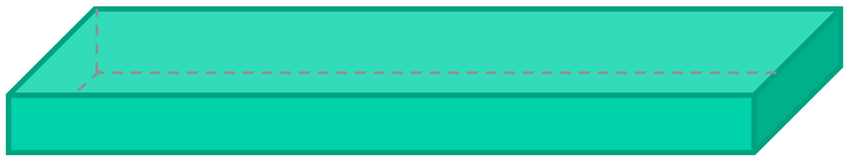
Why?

Suppose that x measures larger than 4 as shown to the right



It wouldn't make sense for two squares to add up to 8 or more inches.

You are making an open top rectangular box



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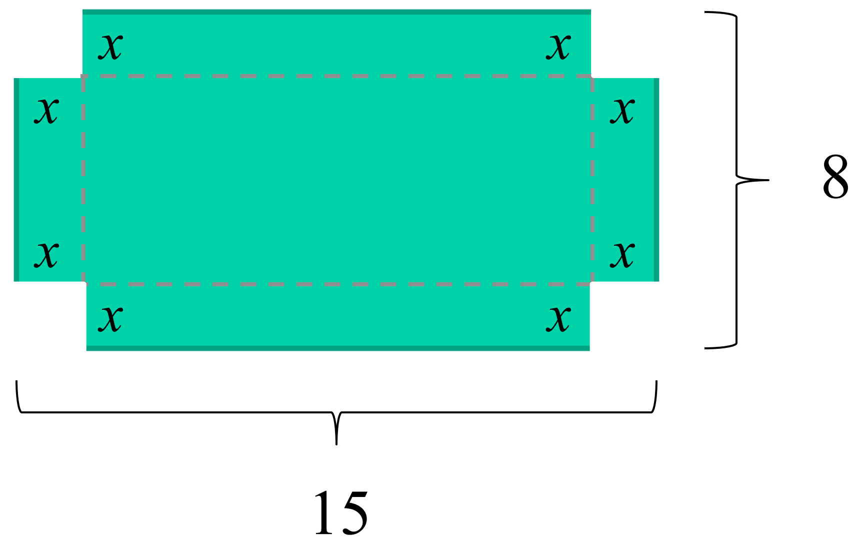
Each square that we cut off would have sides equal to x

$0 < x < 4$ Why?

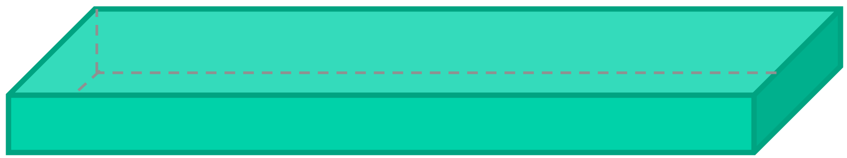


$$V = l \cdot w \cdot h$$

length width height

Now we just need a volume formula in terms of x

You are making an open top rectangular box



Find the dimensions of such a box with the largest volume.

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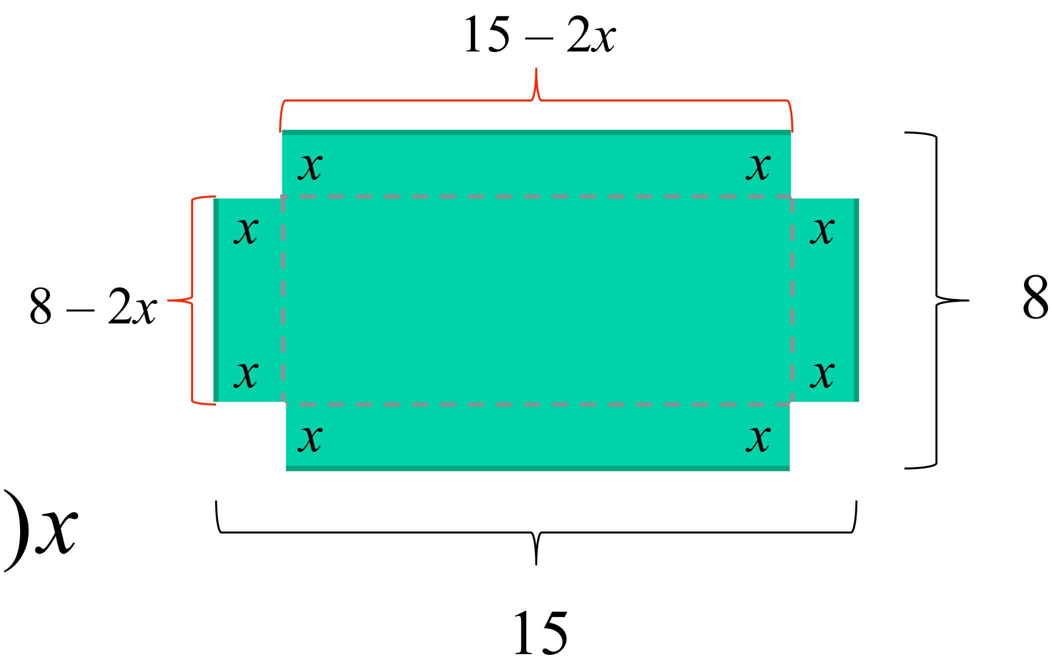
$$V = l \cdot w \cdot h$$

$15-2x$ $\quad$ $8-2x$ $\quad$ x



$$V = (15-2x)(8-2x)x$$

$$V=(15-2x)(8-2x)x$$

$$V=120x-46x^2+4x^3$$

To maximize the volume, we can find any critical points and use them to find the maximum.

$$V'=120-92x+12x^2=0$$

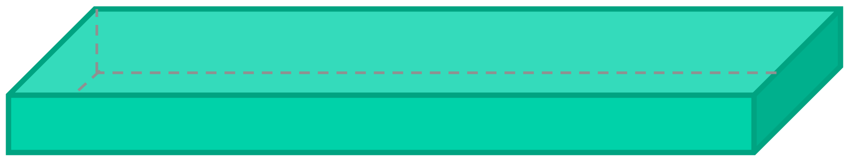
A little factoring produces $x=\frac{5}{3},6$

But we know that 6 is not in the domain for x and both endpoints of x give us a volume of 0

So just double check to be sure that $\frac{5}{3}$

is the x coordinate of a maximum value for $V(x)$.

You are making an open top rectangular box



Find the dimensions of such a box with the largest volume.

from an 8 by 15 inch piece of cardboard

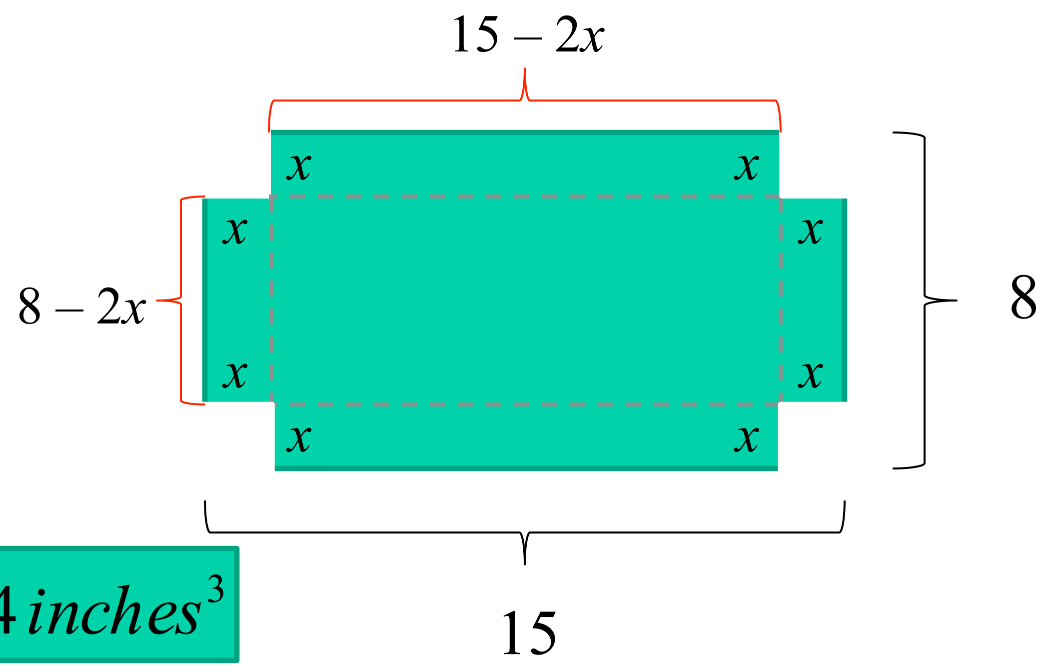
So our dimensions are:

$$l = 15 - 2\left(\frac{5}{3}\right) = \boxed{\frac{35}{3}}$$

$$w = 8 - 2\left(\frac{5}{3}\right) = \boxed{\frac{14}{3}}$$

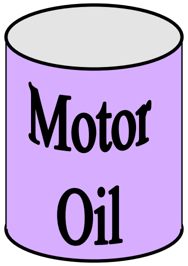
$$h = \boxed{\frac{5}{3}}$$

$$V\left(\frac{5}{3}\right) \approx \boxed{90.74\, inches^3}$$



Example 5:

What dimensions for a one liter cylindrical can will use the <u>least</u> amount of material?



We can <u>minimize</u> the material by minimizing the surface area.

$$A = \underbrace{2\pi r^2}_{\text{area of ends}} + \underbrace{2\pi r h}_{\text{lateral area}}$$

We need another equation that relates r and h:

$$V = \pi r^2 h$$

$$\left(1\ \mathrm{L} = 1000\ \mathrm{cm}^3\right)$$

$$1000 = \pi r^2 h$$

$$\frac{1000}{\pi r^2} = h$$

$$A = 2\pi r^2 + 2\cancel{\pi}\cancel{r} \cdot \frac{1000}{\cancel{\pi} r^{\cancel{2}}}$$

$$A = 2\pi r^2 + \frac{2000}{r}$$

$$A' = 4\pi r - \frac{2000}{r^2}$$

Example 5: What dimensions for a one liter cylindrical can will use the least amount of material?

$$V = \pi r^2 h$$

$$\left(1\ \mathrm{L} = 1000\ \mathrm{cm}^3\right)$$

$$1000 = \pi r^2 h$$

$$\frac{1000}{\pi r^2} = h$$

$$\frac{1000}{\pi (5.42)^2} \approx h$$

$$\boxed{h \approx 10.83\ \mathrm{cm}}$$

$$A = \underbrace{2\pi r^2}_{\text{area of ends}} + \underbrace{2\pi r h}_{\text{lateral area}}$$

$$A = 2\pi r^2 + 2\cancel{\pi}\cancel{r} \cdot \frac{1000}{\cancel{\pi} r^{\cancel{2}}}$$

$$A = 2\pi r^2 + \frac{2000}{r}$$

$$A' = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2}$$

$$\frac{2000}{r^2} = 4\pi r$$

$$2000 = 4\pi r^3$$

$$\frac{500}{\pi} = r^3$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$\boxed{r \approx 5.42\ \mathrm{cm}}$$

Notes and reminders:

If the function that you want to optimize has more than one variable, use substitution to rewrite the function in terms of just one variable.

If you are not sure that the extreme you’ve found is a maximum or a minimum, you have to check.

If the end points could be the maximum or minimum, you have to check.

Remember these examples when working on Assignment 3-4