Logistic Growth Model

The exponential model for population growth($y = y_0e^{kt}$) assumes unlimited growth. This is realistic only for a short period of time when the initial population is small. A more realistic assumption is that the relative growth rate is positive but decreases as the population increases due to environmental or economic factors. In other words there is a maximum population M, the carrying capacity, that the environment is capable of sustaining in the long run. This growth rate can be given by the differential equation:

$$\frac{dP}{dt} = \frac{k}{M}P(M - P) \quad \text{and can also be written as} \quad \frac{dP}{dt} = kP(1 - \frac{P}{M})$$

where P represents the function P(t), M is the carrying capacity, and k is a positive constant. The solution to this **logistic differential equation** is called the **logistic growth model**. Notice that the rate of growth is proportional to both P and (M – P). If P were to exceed M, the growth rate would be negative and the population would be decreasing.

State the solution to the differential equation above.

$$P(t) = \frac{M}{1 + Ae^{-Kt}}$$

A is a constant determined by an initial value

Example:

A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is $\frac{dP}{dt} = 0.0004P(250 - P)$, where time t is in years.

(a) State the formula for the gorilla population in terms of t

$$\frac{K}{M} = \frac{K}{250} = 0.0004$$

$$k = 0.1$$

$$P(t) = \frac{250}{1 + \frac{111}{14}e^{-0.1t}}$$

$P(0) = 28$

(b) about how long will it take for the gorilla population to reach the carrying capacity of the preserve?

$P(t) > 249.5$ since P will never reach 250

$t \approx 82.83$ or about 83 years

1. A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$ where y is the proportion of the population($0 \le y \le 1$) that has heard the rumor at time t.
   (a) What proportion of the population has heard the rumor when it is spreading the fastest?
   (b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.
   (c) At what time t is the rumor spreading the fastest?

$M = 1 \quad K = 2 \quad$ so $\quad y = \frac{1}{1 + Ae^{-2t}}$

a) $\frac{d^2y}{dt^2} = 0 \Rightarrow \frac{dy}{dt} = 2y - 2y^2 \Rightarrow \frac{d^2y}{dt^2} = 2\frac{dy}{dt} - 4y\frac{dy}{dt}$

$\Rightarrow = 2(2y(1-y)) - 4y(2y(1-y)) = 4y - 4y^2 - 8y^2 + 8y^3$

$\Rightarrow = 4y - 12y^2 + 8y^3 = 4y(1 - 3y + 2y^2) = 4y(y-1)(2y-1) = 0$

$y = 0, \frac{1}{2}, 1$

Answer: the rumor is spreading fastest when half the population has heard it $\Leftarrow$ $y = \frac{1}{2}$

b) $y = 0.1$ when $t = 0$ $\quad 0.1 = \frac{1}{1 + Ae^0} \Rightarrow 1 + A = 10 \quad A = 9$

$\boxed{y = \frac{1}{1 + 9e^{-2t}}}$

c) $0.5 = \frac{1}{1 + 9e^{-2t}}$ solve for t w/ the calculator

$t = 1.0986$

2. Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x, diffusion, *dx/dt*, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the differential equation $\frac{dx}{dt} = kx(N-x)$ where N is the size of the population. Suppose that t is time in days, $k = \frac{1}{250}$, and two people start a rumor at time $t = 0$ in a population of 1000 people.

a) Find x as a function of t.

b) When will half the population have heard the rumor?

c) Show that the rumor is spreading the fastest at the time found in (b).

a) To match the original model, we will replace k with $\frac{K}{N} = \frac{1}{250} = \frac{K}{1000}$

$K = 4$

$x(0) = 2 \Rightarrow \quad x(t) = \frac{1000}{1+Ae^{-4t}}, \quad x(0) = 2 = \frac{1000}{1+A}$

$A = 499$

$$x(t) = \frac{1000}{1+499e^{-4t}}$$

c) $\frac{d^2x}{dt^2} = k(Nx' - 2xx') = k^2(N^2x - Nx^2 - 2x(Nx - x^2))$

$\Rightarrow = k^2(N^2x - Nx^2 - 2Nx^2 + 2x^3) = k^2(N^2x - 3Nx^2 + 2x^3)$

$\Rightarrow = k^2x(N^2 - 3Nx + 2x^2) = kx^2(2x-N)(x-N) = 0 \Rightarrow x = 0, \frac{N}{2}, N$

this is true for all Logistic Growth problems

b) $500 = \frac{1000}{1+499e^{-4t}} \Rightarrow t = 1.553$ days

3. Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. the population $P(t)$ is increasing at a rate directly proportional to $800 - P$, where the constant of proportionality is k. (Note that this does not fit the general formula for logistic growth)

a) If $P(0) = 500$, find $P(0)$ in terms of t and k.

b) If $P(2) = 700$, find k.

c) Find $\lim\limits_{t \to \infty} P(t)$

a) $\frac{dP}{dt} = K(800-P) \Rightarrow \frac{dP}{800-P} = K\,dt \Rightarrow \int \frac{dP}{800-P} = \int K\,dt$

$\Rightarrow -\ln|800-P| = Kt + C \Rightarrow 800 - P = Ae^{-Kt} \Rightarrow 800 - Ae^{-Kt} = P$

$500 = 800 - Ae^{0} \Rightarrow A = 300 \Rightarrow P(t) = 800 - 300e^{kt}$

b) $P(2) = 700 = 800 - 300e^{K(2)} \Rightarrow 300e^{K(2)} = 100 \Rightarrow e^{K(2)} = \frac{1}{3}$

$\Rightarrow 2K = -\ln 3 \Rightarrow K = -\frac{\ln 3}{2}$

c) $\lim\limits_{t \to \infty} 800 - 300e^{-\frac{\ln 3}{2}t} = 800$